Mathematical Statistical Physics – LMU München, summer semester 2012

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Homework sheet 10 – due on 29.06.2012 – and exercises for the class on 22.06.2012

10.1 *DLR condition and conditional probability.* Let $\mu_\Lambda^{\beta,\beta'}(.|\eta)$ denote the grand canonical measure of some system in the box Λ with the boundary condition $\eta \in \Omega_{\Lambda^c}$. Using this, define

$$\gamma_\Lambda^{\beta,\beta'}(.|\eta) := \mu_\Lambda^{\beta,\beta'}(.|\eta \cap \Lambda^c) \otimes \delta_{\eta\cap\Lambda^c}$$

for all $\eta \in \Omega$ as a product measure on $\Omega = \Omega_\Lambda \times \Omega_{\Lambda^c}$.

The Dobrushin-Lanford-Ruelle condition for a measure μ on Ω to be Gibbs is

$$\mu = \mu \otimes \gamma_\Lambda^{\beta,\beta'} \quad \text{for every bounded } \Lambda,$$

where $\gamma_\Lambda^{\beta,\beta'}$ is viewed as a probability kernel from Ω to Ω. Show that this is the same as requiring that

$$\mu_1 = \mu_2 \otimes \mu_\Lambda^{\beta,\beta'},$$

where μ_1 and μ_2 are the two marginals of the measure μ on $\Omega = \Omega_\Lambda \times \Omega_{\Lambda^c}$, and $\mu_\Lambda^{\beta,\beta'}$ is viewed as a probability kernel from Ω_{Λ^c} to Ω_Λ.

10.2 **(homework)** *The partition function and the effect of ignoring the velocity.* Consider a system of interacting point particles in d dimensions with Hamiltonian $H(q,p) = \sum_i \frac{\vec{p}_i}{2m} + \sum_{i<j} \Phi(|q_i - q_j|)$. Consider the canonical partition function $Z^{can}(V,N,\beta)$ and the grand canonical partition function $Z^{gr}(V,\beta,\beta')$ (or $Z^{gr}(V,\beta,z)$ if you like), which are integrals of some weight functions on the phase space.

Now define the *configurational partition functions* by "ignoring the velocity". That is, e.g. $Z^{can}_{conf}(V,N,\beta)$ is an integral on the *configuration space only* of the canonical weight function with the kinetic energy omitted.

(a) Find the relation between the partition function and the configurational partition function, both in the canonical and the grand canonical setting.

(b) Calculate the density of the free gas with parameters β, z and m.

10.3 **(homework)** *Is the grand canonical ensamble with boundary condition well defined?* Show that the grand canonical measure is well defined with any boundary condition in a system of interacting point particles with a bounded finite range pair interaction. What can we say if the pair interaction is only tempered and stable?

10.4 *Consistency property.* Show that $\gamma_{\Lambda'}^{\beta,\beta'} = \gamma_{\Lambda'}^{\beta,\beta'} \otimes \gamma_\Lambda^{\beta,\beta'}$ for every bounded $\Lambda \subset \Lambda' \subset \mathbb{R}^d$.

10.5 *Gibbs measures of the free gas.* Find and describe every Gibbs measure of the free gas.

10.6 **(homework)** *Ising model in one dimension.* For the Ising model in one dimension, let the phase space be $\omega = \{-1,1\}^N$ and the Hamiltonian be $H : \Omega \to \mathbb{R}$ be defined as

$$H(\sigma_1,\ldots,\sigma_N) := \sum_{i=1}^N (J\sigma_i\sigma_{i+1} + h\sigma_i).$$

(We use the convention $\sigma_{N+1} := \sigma_1$, which corresponds to periodic boundary conditions.)

(a) Calculate the partition function

$$Z(N, \beta, h) := \sum_{\sigma \in \Omega} e^{-\beta H(\sigma)}.$$

Hint: In the expression definig Z, discover a power of a 2×2 matrix. If you do it well, this matrix will be symmetric. In the end, you only need to calculate the eigenvalues.

(b) Calcualte the entropy in the thermodynamic limit (understand the question well) and show that it is a smooth function of h for every $\beta > 0$.