Mathematical Statistical Physics – LMU München, summer semester 2012
Hartmut Ruhl, Imre Péter Tóth
Exercises for the class on 20.07.2012

14.1 **Definition 1** *The endomorphisms* $(M_1, \mathcal{F}_1, \mu_1, T_1)$ *and* $(M_1, \mathcal{F}_1, \mu_1, T_1)$ *are said to be* isomorphic *or* conjugate, *M_1 and M_2 can be identified (apart from sets of zero measure) in such a way, that the measure and the dynamics are also transferred from one to the other and back.*

That is, there exist $M_1' \subset M_1$ and $M_2' \subset M_2$ with $\mu_1(M_1') = \mu_2(M_2') = 1$ and a $\varphi : M_i' \to M_2'$ measureable and invertable with a measurable inverse, such that

- *for any $A_1 \in \mathcal{F}_1$ with $A_1 \subset M_1'$, $\mu_2(\varphi(A_1)) = \mu_1(A_1)$*
- *for any $A_2 \in \mathcal{F}_2$ with $A_2 \subset M_2'$, $\mu_1(\varphi^{-1}(A_2)) = \mu_2(A_2)$*
- *$\varphi \circ T_1 = T_2 \circ \varphi$ (on M_1')*
- *$\varphi^{-1} \circ T_2 = T_1 \circ \varphi^{-1}$ (on M_2')*

(a) Show that being conjugate is an equivalence realtion on endomophisms.

(b) Show that being ergodic is a conjugacy-invariant property.

(c) Show that being mixing is a conjugacy-invariant property.

14.2 Show that the doubling map $T : [0, 1] \to [0, 1]$ defined as

$$T(x) := \begin{cases} 2x & \text{if } x < 1/2, \\ 2x - 1 & \text{if } x \geq 1/2, \end{cases}$$

with Lebesgue meaasure on $[0, 1]$, is conjugate to the one-sided Bernoulli (left) shift on $0, 1^{\mathbb{N}}$ with the measure which is the product of $Uni(0, 1)$ on the coordinates. Conclude that the doubling map is ergodic and mixing.

14.3 Show that the baker's map $T : [0, 1] \times [0, 1] \to [0, 1] \times [0, 1]$ defined as

$$T((x, y)) := \begin{cases} (2x, y/2) & \text{if } x < 1/2, \\ (2x - 1, (y + 1)/2) & \text{if } x \geq 1/2, \end{cases}$$

with Lebesgue meaasure on $[0, 1] \times [0, 1]$, is conjugate to the full Bernoulli (left) shift on $0, 1^{\mathbb{Z}}$ with the measure which is the product of $Uni(0, 1)$ on the coordinates. Conclude that the baker's map is ergodic and mixing.

14.4 *A toy example of renormalization.* Let $\mathcal{M}$ be the set of functions $\psi : \mathbb{R} \to \mathbb{C}$ with the following properties:

- ψ is continuous,
- ψ is twice differentiable at 0,
- $\psi'(0) = 0$.

(a) Find a map $RG : \mathcal{M} \to \mathcal{M}$ with the property that if X_1 and X_2 are real valued i.i.d. random variables with characteristic function $\psi \in \mathcal{M}$, then $RG\psi$ is the characteristic function of $\frac{X_1+X_2}{\sqrt{2}}$.

(b) What are the fixed points of RG?

(c) Let ψ_0 be the characteristic function of $Uni(-1, 1)$. What is the meaning of $RG^n\psi_0$?

(d) How is $(RG\psi)''(0)$ related to $\psi''(0)$?

(e) Show that $RG^n\psi_0$ converges and find the limit.