Mathematical Statistical Physics – LMU München, summer semester 2012
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Homework sheet 12 – solutions

12.1 **(homework)** *Ising model: Weak sensitivity of the canonical probability of a "+" on a single neighbouring spin at high temperature.* Consider the (nearest neighbour ferromagnetic) Ising model without external field on $\mathbb{Z}^d$, with interaction constant $J = 1$:

$$H(\sigma) = -\frac{1}{2}\sum_{|i-j|=1} \sigma_i\sigma_j = -\sum_{\{i,j\}:i\sim j} \sigma_i\sigma_j$$

(That is: count every pair of neighbours once with interaction 1, or count every pair twice with interaction 1/2.) Let $\gamma_k^\beta(+|\eta)$ denote the probability of finding a spin +1 at position $k \in \mathbb{Z}^d$ with respect to the canonical measure in the box $\Lambda = \{k\}$ with boundary condition $\eta \in \Omega = \{-1,1\}^{\mathbb{Z}^d}$. For $\eta \in \Omega$ and $j \in \mathbb{Z}^d$, let η_j^+ denote the configuration which is the same as η everywhere except possibly at j, where it is +1. Define η_j^- similarly.

Show that for any $k, j \in \mathbb{Z}^d$ and any $\eta \in \Omega$,

$$|\gamma_k^\beta(+|\eta_j^+) - \gamma_k^\beta(+|\eta_j^-)| \le \beta \mathbf{1}_{k\sim j}.$$

Solution: Since the interaction is nearest-neighbour, $\gamma_k^\beta(+|\eta)$ clearly depends only on the spins neighbouring k (i.e. also not on the spin at k), so $\gamma_k^\beta(+|\eta_j^+) - \gamma_k^\beta(+|\eta_j^-) = 0$ unless $k \sim j$. Moreover, $\gamma_k^\beta(+|\eta)$ depends only on the number of + spins around k – which can be $0, 1, \ldots, 2d$. Let $\gamma(m)$ denote the value of $\gamma_k^\beta(+|\eta)$ for those η for which this number is m. This way if $j \sim k$,

$$|\gamma_k^\beta(+|\eta_j^+) - \gamma_k^\beta(+|\eta_j^-)| = |\gamma(m+1) - \gamma(m)|$$

for some $m \in \{0, 1, \ldots, 2d-1\}$ and we only need to find the maximum of these $2d$ values. For that, we explicitely calculate $\gamma(m)$. First calculate the energies $H(+|m) := H_k(+|\eta)$ and $H(-|m) := H_k(-|\eta)$ when η has m "+" spins around k:

$$\begin{aligned} H(+|m) &= -[m \cdot ((+1)\cdot(+1)) + (2d-m)\cdot((+1)\cdot(-1))] = 2(d-m) \\ H(-|m) &= -[m \cdot ((-1)\cdot(+1)) + (2d-m)\cdot((-1)\cdot(-1))] = 2(m-d) \end{aligned}$$

so

$$\gamma(m) = \frac{e^{-\beta H(+|m)}}{e^{-\beta H(+|m)} + e^{-\beta H(-|m)}} = \frac{e^{2\beta(m-d)}}{e^{2\beta(m-d)} + e^{2\beta(d-m)}}$$

(increasing in m, of course, since the model is ferromagnetic). We simply estimate the largest possible value of $\gamma(m+1) - \gamma(m)$ by the maximal derivative of $\gamma(x)$ (to be precise, we can refer to the Lagrange mean value theorem):

$$\gamma'(x) = 4\beta\frac{e^{4\beta(d-x)}}{(1+e^{4\beta(d-x)})^2}.$$

If we introduce $y := e^{4\beta(d-x)}$, we get

$$\gamma'(x) = 4\beta\frac{y}{(1+y)^2} \le 4\beta\frac{1}{4} = \beta$$

which implies

$$|\gamma(m+1) - \gamma(m)| = \gamma(m+1) - \gamma(m) \le \beta.$$

12.2 **(homework)** *Ising model: independence of the thermodynamic limit from the boundary conditions.*

(a) Consider a finite $\Lambda \subset \mathbb{Z}^d$ and the phase space $\Omega_\Lambda = \{-1, 1\}^\Lambda$ with two different Hamiltonians $H_\Lambda, H'_\Lambda : \Omega \to \mathbb{R}$. Let $||H_\Lambda - H'_\Lambda||$ denote the distance of H_Λ and H'_Λ in the supremum norm:

$$||H_\Lambda - H'_\Lambda|| := \max_{\sigma \in \Omega_\Lambda} |H_\Lambda(\sigma) - H'_\Lambda(\sigma)|.$$

Consider the canonical distributions with the partition functions $Z(\Lambda, \beta)$ and $Z'(\Lambda, \beta)$ and, as usual, let $p_\Lambda(\beta)$ and $p'_\Lambda(\beta)$ denote the thermodynamic pressures for the two Hamiltonians:

$$\begin{aligned} p_\Lambda(\beta) &= \frac{1}{\beta|\Lambda|} \log Z(\Lambda, \beta), \\ p'_\Lambda(\beta) &= \frac{1}{\beta|\Lambda|} \log Z'(\Lambda, \beta). \end{aligned}$$

Estimate $|p_\Lambda(\beta) - p'_\Lambda(\beta)|$ with the help of $||H_\Lambda - H'_\Lambda||$.

(b) We say that $\Lambda_n \nearrow \mathbb{R}^d$ if $\Lambda_n \subset \Lambda_{n+1}$ and $\bigcup_n \Lambda_n = \mathbb{R}^d$. We say that $\Lambda_n \nearrow \mathbb{R}^d$ in the sense of van Hove if $\Lambda_n \nearrow \mathbb{R}^d$ and $\frac{|\partial\Lambda_n|}{|\Lambda_n|} \to 0$, where $\partial\Lambda$ denotes the boundray of Λ:

$$\partial\Lambda := \{k \in \Lambda : dist(k, \Lambda^c) = 1\}.$$

Consider the Ising model (with external filed) for a sequence Λ_n such that $\Lambda_n \nearrow \mathbb{R}^d$ in the sense of van Hove, and with an arbitrary sequence η_n of boundary conditions. Show that the limiting thermodynamic pressure

$$p(\beta, h) := \lim_{n\to\infty} p_{\Lambda_n}(\beta, h)$$

is independent of the boundary conditions.

(Remark: We have not fully checked, but it is true that the limit exist for empty boundary conditions, and is independent of the sequence Λ_n.)

Solution:

(a) To write less, introduce $K := ||H_\Lambda - H'_\Lambda||$. With this, for every $\sigma \in \Omega_\Lambda$

$$e^{-\beta K} \le \frac{H'_\Lambda(\sigma)}{H_\Lambda(\sigma)} \le e^{\beta K},$$

which implies

$$e^{-\beta K} \le \frac{Z'(\Lambda, \beta)}{Z(\Lambda, \beta)} \le e^{\beta K},$$

which in turn implies

$$-\frac{K}{|\Lambda|} \le p'_\Lambda(\beta) - p_\Lambda(\beta) \le \frac{K}{|\Lambda|}.$$

that is,

$$|p_\Lambda(\beta) - p'_\Lambda(\beta)| \le \frac{||H_\Lambda - H'_\Lambda||}{|\Lambda|}.$$

(b) Let η_n and ν_n be any two boundary conditions and let

$$\begin{aligned} H_{\Lambda_n}(\sigma) &:= H_{\Lambda_n}(\sigma|\eta_n), \\ H'_{\Lambda_n}(\sigma) &:= H_{\Lambda_n}(\sigma|\nu_n). \end{aligned}$$

Since the interaction is nearest neighbour and bounded, the contribution of the boundary condition to the energy is limited by the number of spins at the boundary:

$$||H_\Lambda - H'_\Lambda|| \leq const|\partial\Lambda_n|.$$

By the previous part this implies

$$|p'_{\Lambda_n}(\beta,h) - p_{\Lambda_n}(\beta,h)| \leq const\frac{|\partial\Lambda_n|}{|\Lambda_n|}$$

which goes to zero by our assumption that $\Lambda_n \nearrow \mathbb{R}^d$ in the sense of van Hove.

12.3 *Symmetry braking in the 2d Ising model: Peierls argument.* For $\mathbb{Z}^2$ an *elementary contour* γ is a sequence of *edges* of $\mathbb{Z}^2$ which forms a non-self-intersecting circle of the dual graph. Such a contour has an "inside" and an "outside", both consisting of points of $\mathbb{Z}^2$. Consider the (nearest neighbour) ferromagnetic Ising model on $\mathbb{Z}^2$ without external field. For a configuration $\sigma \in \Omega = \{\}^{\mathbb{Z}^d}$, an elementary contour γ is said to be *present* if all the spins adjacent to the contour on the inside are $+1$ and all the spins adjacent to the contour on the outside are -1, or vice versa.

(a) For $\Lambda \subset \mathbb{R}^2$ and an elementary contour γ which is "inside Λ", show that there is a $1-1$ correspondance between configurations σ for which γ is present, and configurations σ' for which all the spins adjacent to γ are the same.

(b) Estimate the probability (w.r.t. the canonical measure) of a given contour γ of length n being present.

(c) Give a rough estimate for the number of elementary contours of length n that surrond the origin. The estimate can be rough, but make sure that it's at most exponential in n.

(d) Based on the previous two points, give the easiest possible estimate for the probability that a contour surrounding the origin is present.

(e) Consider the ferromagnetic Ising model (without external field) in the box $\Lambda \subset \mathbb{R}^d$ with the boundary condition "$\eta = +1$ everywhere". Show that if β is big enough, then there is an $\alpha = \alpha(\beta) < \frac{1}{2}$ such that for *any* Λ

$$\mu^\beta_{can}(\{\sigma_0 = -1\}|\eta) \leq \alpha.$$

(f) Conclude that for β big enough, the Gibbs measure is not unique.

(g) Conclude that for β big enough, $p(\beta,h)$ is not differentiable in h at $h=0$.