Probability 1
CEU Budapest, fall semester 2017
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Homework sheet 1 – due on 03.10.2017

1. Define a σ-algebra as follows:

   Definition 1 *For a nonempty set Ω, a family $\mathcal{F}$ of subsets of ω (i.e. $\mathcal{F} \subset 2^{\Omega}$, where $2^{\Omega} := \{A : A \subset \Omega\}$ is the power set of Ω) is called a σ-algebra over Ω if*

   - $\emptyset \in \mathcal{F}$
   - *if $A \in \mathcal{F}$, then $A^C := \Omega \setminus A \in \mathcal{F}$ (that is, $\mathcal{F}$ is closed under complement taking)*
   - *if $A_1, A_2, \dots \in \mathcal{F}$, then $(\cup_{i=1}^{\infty} A_i) \in \mathcal{F}$ (that is, $\mathcal{F}$ is closed under countable union).*

   Show from this definition that a σ-algebra is closed under countable intersection, and under finite union and intersection.

2. *Continuity of the measure*

   (a) Prove the following:

   Theorem 1 *(Continuity of the measure)*

   i. If $(\Omega, \mathcal{F}, \mu)$ is a measure space and $A_1, A_2, \dots$ is an increasing sequence of measurable sets (i.e. $A_i \in \mathcal{F}$ and $A_i \subset A_{i+1}$ for all i), then $\mu(\cup_{i=1}^{\infty} A_i) = \lim_{i\to\infty} \mu(A_i)$ (and both sides of the equation make sense).

   ii. If $(\Omega, \mathcal{F}, \mu)$ is a measure space, $A_1, A_2, \dots$ is a decreasing sequence of measurable sets (i.e. $A_i \in \mathcal{F}$ and $A_i \supset A_{i+1}$ for all i) and $\mu(A_1) < \infty$, then $\mu(\cap_{i=1}^{\infty} A_i) = \lim_{i\to\infty} \mu(A_i)$ (and both sides of the equation make sense).

   (b) Show that in the second statement the condition $\mu(A_1) < \infty$ is needed, by constructing a counterexample for the statement when this condition does not hold.

3. **(homework)**

   (a) We toss a biased coin, on which the probability of heads is some $0 \le p \le 1$. Define the random variable ξ as the indicator function of tossing heads, that is

   $$\xi := \begin{cases} 0, & \text{if tails} \\ 1, & \text{if heads} \end{cases} .$$

   i. Describe the distribution of ξ (called the Bernoulli distribution with parameter p) in the "classical" way, listing possible values and their probabilities,

   ii. and also by describing the distribution as a measure on $\mathbb{R}$, giving the weight $\mathbb{P}(\xi \in B)$ of every (Borel) subset B of $\mathbb{R}$.

   iii. Calculate the expectation of ξ.

   (b) We toss the previous biased coin n times, and denote by X the *number of heads* tossed.

   i. Describe the distribution of X (called the Binomial distribution with parameters (n, p)) by listing possible values and their probabilities.

   ii. Calculate the expectation of X by the old "probability 1" definition, using its distribution,

   iii. and also by noticing that $X = \xi_1 + \xi_2 + \dots + \xi_n$, where ξ_i is the indicator of the i-th toss being heads, and using linearity of the expectation.

4. **(homework)** *Usefulness of the linearity of the expectation.* A building has 10 floors, not including the ground floor. On the ground floor, 10 people get into the elevator, and every one of them chooses a destination at random, uniformly out of the 10 floors, independently of the others. Let X denote the number of floors *on which the elevator stops* – i.e. the number of floors that were chosen by at least one person. Calculate the expectation of X. *(Hint: First notice that the distribution of X is hard to calculate. Find a way to calculate the expectation without that. Help: What is the probability that the elevator stops on the first floor?)*

5. **(homework)** We take a huge bag. 1 minute before midnight we put 10 balls (numbered $1 \dots 10$) into the bag. Then we draw a ball from the bag at random, and throw it away. $\frac{1}{2}$ minute before midnight we put another 10 balls (numbered $11 \dots 20$) into the bag. Then we draw a ball from the bag at random, and throw it away. $\frac{1}{4}$ minute before midnight we put another 10 balls (numbered $21 \dots 30$) into the bag. Then we draw a ball from the bag at random, and throw it away. And so on, infinitely many times: $\frac{1}{2^n}$ minute before midnight we put 10 balls (numbered $(10n+1) \dots (10n+10)$) into the bag. Then we draw a ball from the bag at random, and throw it away.

   a.) What is the probability that ball number 1 will be in the bag at midnight? *(Hint: we will see later that* $\lim_{N\to\infty} \prod_{n=0}^{N} \left(1 - \frac{1}{9n+10}\right) = 0$*.)*

   b.) What is the probability that ball number 11 will be in the bag at midnight?

   c.) Show that, at midnight, with probability 1, the bag will be empty. (What?!?!)