

Probability 1
CEU Budapest, fall semester 2017

Imre Péter Tóth

Homework sheet 5 – due on 28.11.2017 – and exercises for practice

5.1 (**homework**) Consider the probability space $\Omega = \{a, b, c\}$ equipped with the uniform measure as \mathbb{P} (so $\mathbb{P}(\{a\}) = \mathbb{P}(\{b\}) = \mathbb{P}(\{c\}) = \frac{1}{3}$). Let the random variable $X : \Omega \rightarrow \mathbb{R}$ be such that $X(a) = X(b) = 0, X(c) = 1$.

a.) Let D_1 be the partition $\{\{a\}, \{b, c\}\}$. Find the conditional expectation $\mathbb{E}(X|D_1)$ (which is the same as $\mathbb{E}(X|G_1)$, where the σ -algebra G_1 is $G_1 = \{\emptyset, \{a\}, \{b, c\}, \Omega\}$.)

b.) Let D_2 be the partition $\{\{a, b\}, \{c\}\}$. Find the conditional expectation $\mathbb{E}(X|D_2)$ (which is the same as $\mathbb{E}(X|G_2)$, where the σ -algebra G_2 is $G_2 = \{\emptyset, \{a, b\}, \{c\}, \Omega\}$.)

5.2 Durrett [1], Exercise 5.1.6

5.3 Durrett [1], Exercise 5.2.3

5.4 Durrett [1], Exercise 5.2.4

5.5 Let X_n be a martingale w.r.t. the filtration \mathcal{F}_n on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let the random variable $\tau : \Omega \rightarrow \mathbb{N}$ be a *stopping time*, meaning

$$\{\tau = k\} := \{\omega \in \Omega \mid \tau(\omega) = k\} \in \mathcal{F}_k \quad \text{for every } k.$$

Using the notation $a \wedge b := \min\{a, b\}$, we introduce the process

$$Y_n := X_{\tau \wedge n} = \begin{cases} X_n & \text{if } n < \tau, \\ X_\tau & \text{if } n \geq \tau. \end{cases}$$

Show that Y_n is also a martingale w.r.t. \mathcal{F}_n . (Hint: Y_n is the fortune of a gambler with a certain strategy.)

5.6 (**homework**) Let $p \in (0, 1)$ be fixed, and let $q = 1 - p$. A frog performs a (discrete time) random walk on the 1-dimensional lattice \mathbb{Z} the following way:

The initial position is $X_0 = 0$. The frog jumps 1 step up with probability p and jumps 1 step down with probability q at each time step, independently of what happened before, until it reaches either the point $a = -10$ or the point $b = +30$, which are *sticky*: if the frog reaches one of them, it stays there forever.

Let X_n denote the position of the frog after n steps (for $n = 0, 1, 2, \dots$).

a.) Show that $Y_n := \left(\frac{q}{p}\right)^{X_n}$ is a martingale (w.r.t. the natural filtration).

b.) Show that Y_n converges almost surely to some limiting random variable Y_∞ . What are the possible values of Y_∞ ?

c.) How much is $\mathbb{E}Y_\infty$ and why?

d.) Suppose now that $p \neq \frac{1}{2}$. Use the previous results to calculate the probability that the frog eventually gets stuck at the point $a = -10$.

5.7 Let $0 \leq p \leq 1$ and $q = 1 - p$. Let X_1, X_2, \dots be i.i.d. with $\mathbb{P}(X_i = -1) = q$ and $\mathbb{P}(X_i = 1) = p$. For $n = 0, 1, \dots$ let $S_n = X_1 + \dots + X_n$. So S_n is a simple asymmetric random walk starting from $S_0 = 0$. (Symmetric if $p = \frac{1}{2}$.) Show that $M_n := S_n - n(p - q)$ is a martingale (w.r.t. the natural filtration).

For $p \neq q$, use this to find the expectation of the time when the frog of the previous exercise gets stuck.

5.8 Let X_1, X_2, \dots be i.i.d. with $\mathbb{P}(X_i = -1) = \mathbb{P}(X_i = 1) = \frac{1}{2}$. For $n = 0, 1, \dots$ let $S_n = X_1 + \dots + X_n$. So S_n is a simple symmetric random walk starting from $S_0 = 0$.

a.) Show that $S_n^2 - n$ is a martingale (w.r.t. the natural filtration). *This is a special case of Durrett [1], Exercise 5.2.6. You can also solve that – it’s not any harder.*

b.) Use this to find the expectation of the stopping time when the walk first reaches either -10 or 30 .

c.) How about the expectation of the stopping time when the walk first reaches 30 ?

5.9 Let $0 \leq p \leq 1$ and $q = 1 - p$. Let X_1, X_2, \dots be i.i.d. with $\mathbb{P}(X_i = -1) = q$ and $\mathbb{P}(X_i = 1) = p$. For $n = 0, 1, \dots$ let $S_n = X_1 + \dots + X_n$. So S_n is a simple asymmetric random walk starting from $S_0 = 0$. (Symmetric if $p = \frac{1}{2}$.)

a.) Show that $M_n := \left(\frac{q}{p}\right)^{S_n}$ is a martingale (w.r.t. the natural filtration).

b.) Let $H \subset \mathbb{Z}$ and let τ be the random time when the random walk first reaches H , so

$$\tau = \inf\{n \mid S_n \in H\}.$$

Show that $M_{\tau \wedge n}$ is also a martingale.

c.) Let $p = \frac{1}{3}$. What is the probability that the walk ever reaches 10 ?

5.10 *Life, the Universe, and Everything.* Arthur decides to keep rolling a fair die until he manages to roll two 6-es consecutively. What is the expected number of rolls he needs?

5.11 (**homework**) Bob keeps tossing a fair coin and makes notes of the results: he writes “H” for heads and “T” for tails. Calculate the expected number of tosses

a.) until the character sequence “HTHT” shows up,

b.) until the character sequence “THTT” shows up.

5.12 Alice and Bob keep tossing a fair coin until either the word $A := \text{“HTHT”}$ or the word $B := \text{“THTT”}$ shows up. If the word appearing first is A , then Alice wins, and if B , then Bob. Introduce the notation $p_A := \mathbb{P}(\text{Alice wins})$, $p_B := \mathbb{P}(\text{Bob wins})$. Let τ be the random time when the game ends.

a.) Think of a casino, as in the solution of the ABRACADABRA problem [2], where all players bet for (consecutive letters of) the word A . Using the capital of this casino as a martingale, express $\mathbb{E}\tau$ using p_A and p_B .

b.) Now think of another casino, where all players bet for (consecutive letters of) the word B . Using the capital of this other casino as a martingale, get another expression for $\mathbb{E}\tau$ using p_A and p_B .

c.) Solve the system of equations formed by the two equations above, to calculate $\mathbb{E}\tau$, p_A and p_B .

5.13 **Definition 1** *The generating function of a non-negative, integer valued random variable X is the power series $z \rightarrow g_X(z) := \sum_{k=0}^{\infty} \mathbb{P}(X = k)z^k$, which is convergent at least on the $[-1, 1]$ interval.*

If we know g_X , then the distribution of X can be reconstructed by Taylor expansion.

A monkey keeps pressing keys of a typewriter with 26 keys printing the letters of the English alphabet, uniformly and independently of the past, until the word “ABRACADABRA” shows up. Denote this random time by τ . Beside the monkey – as in the original ABRACADABRA solution [2] – operates a casino where players can always bet for the next key pressed in a fair game: if their guess is wrong, they lose their bet entirely, while if it is correct, they lose it and get back 26 times more.

Before every key press, a new player arrives, who will bet all his money first on “A”, then on “B”, then on “R”, etc. through the ABRACADABRA sequence, as long as he keeps winning or the game ends. (If he loses once, he goes home immediately.) This is just like in the original ABRACADABRA solution.

However, the later a player arrives, the less money he has to play with: there is some fixed $z \in (0, 1)$ such that the n -th player arrives with $\$z^{n-1}$.

Show that the fortune of the casino is a martingale, and use the optional stopping theorem to calculate the generating function of τ .

5.14 Let N, X_1, X_2, X_3, \dots be independent, and let them all have (optimistic) geometric distribution with parameter $p = \frac{1}{6}$. Calculate the expectation of

$$S =: \sum_{k=1}^N (X_k + 1).$$

What has this got to do with Exercise 10?

Hint: $\mathbb{E}S = \mathbb{E}(\mathbb{E}(S | N)) = \sum_{n=0}^{\infty} \mathbb{P}(N = n)\mathbb{E}(S | N = n)$.

References

- [1] Durrett, R. *Probability: Theory and Examples*. 4th edition, Cambridge University Press (2010)
- [2] Ai, Di. *Martingales and the ABRACADABRA problem*.
<http://math.uchicago.edu/~may/VIGRE/VIGRE2011/REUPapers/Ai.pdf> (2011)