1) Let $a_n = \frac{1}{\sqrt{n}} \int_0^1\int_0^1 \dots \int_0^1 \sqrt{x_1^2+\dots+x_n^2}\, dx_1 dx_2 \dots dx_n = \int_0^1\int_0^1 \dots \int_0^1 \sqrt{\frac{x_1^2+\dots+x_n^2}{n}}\, dx_1 dx_2 \dots dx_n$.

Then $a_n = \mathbb{E}\sqrt{\frac{X_1^2+\dots+X_n^2}{n}}$ where $X_1, \dots, X_n$ are independent $\sim \mathrm{Uni}[0,1]$.

By the law of large numbers, $Y_n := \frac{X_1^2+\dots+X_n^2}{n} \Rightarrow \mathbb{E}X_1^2 = \int_0^1 x^2\,dx = \frac{1}{3}$.

The function $f: [0,1] \to \mathbb{R}$ $f(x) := \sqrt{x}$ is bounded and continuous,

so $a_n = \mathbb{E}\, f(Y_n) \to \mathbb{E}\, f(\frac{1}{3}) = \underline{\underline{\sqrt{\frac{1}{3}}}}$.

2) 1.) ~~Let~~ Thm a.) If X_n has char. fn φ_n, X has char. fn φ and $X_n \Rightarrow X$, then $\varphi_n(t) \to \varphi(t)$ for every $t \in \mathbb{R}$.

b.) If X_n has char. fn φ_n and $\varphi(t) := \lim_{n\to\infty} \varphi_n(t)$ exists for every $t \in \mathbb{R}$, AND $\varphi = \varphi(t)$ is continuous at 0, ~~then~~ then $\exists$ a random variable X s.t. $X_n \Rightarrow X$ and X has char. fn φ.

2.) Thm (CLT): If $X_1, \dots X_n, \dots$ are i.i.d with $\mathbb{E}X_i = m \in \mathbb{R}$, $\mathrm{Var}\, X_i = \sigma^2 < \infty$, then $\frac{X_1+\dots+X_n - nm}{\sqrt{n}\,\sigma} \Rightarrow N(0,1)$.

Proof: Assume w.l.o.g that $m=0$. Let $S_n = X_1 + \dots + X_n$.

Then $\varphi_{X_1}(t) = 1 + i\,\mathbb{E}X_1 t + \frac{1}{2} i^2\, \mathbb{E}(X_1^2) t^2 + o(t^2) = 1 - \frac{\sigma^2}{2} t^2 + o(t^2)$

so $\varphi_{S_n}(t) = (\varphi_{X_1}(t))^n = \left(1 - \frac{\sigma^2}{2} t^2 + o(t^2)\right)^n$

so $\varphi_{\frac{S_n}{\sqrt{n}\sigma}}(t) = \varphi_{S_n}\left(\frac{t}{\sqrt{n}\,\sigma}\right) = \left[1 + \frac{-t^2/2}{n} + o\left(\frac{1}{n}\right)\right]^n \xrightarrow{n\to\infty} e^{-\frac{t^2}{2}}$, which is exactly the char. fn of $N(0,1)$, so the continuity thm gives the statement. $\square$

3) a.) $\mathbb{E}X_n^2 < \infty$ by assumption.

$\sigma(X_n^2) \subset \sigma(X_n)$, so $X_n^2 \in \mathcal{F}_n$ because $X_n \in \mathcal{F}_n$

$$\mathbb{E}(X_{n+1}^2 \mid \mathcal{F}_n) \underset{\substack{\text{Jensen's}\\ \text{inequality}}}{\geq} \left[\mathbb{E}(X_{n+1} \mid \mathcal{F}_n)\right]^2 \xlongequal[\text{martingale}]{X_n \text{ is a}} X_n^2 \quad \square$$

b.) Let $X_n \equiv -\frac{1}{n}$. Then clearly $\mathbb{E}(X_{n+1} \mid X_n) = -\frac{1}{n+1} > -\frac{1}{n} = X_n$.

But $X_n^2 = \frac{1}{n^2}$ is decreasing.

4) Let X_n be the position of the frog after n steps.
Then X_n is a nonnegative martingale $\xRightarrow{\text{almost surely}}$ it is convergent by the martingale convergence theorem.
Since it is integer valued, it can only be convergent by being eventually constant.
However, the frog can not get stuck at any $k \neq 0$, because

$$\sum_{n \geq n_0} \mathbb{P}(\text{it jumps away}) = \frac{k}{k+1} \sum_{n \geq n_0} \frac{1}{n} = \infty \quad (\text{for } k \neq 0),$$

so the Borel-Cantelli lemma ensures that it would jump away.

Summary: The frog does get stuck, but it can only get stuck at 0 so $\boxed{\mathbb{P}(\text{reaching } 0) = 1.}$

5) Let $\xi_1, \xi_2, \xi_3 \ldots$ be independent, $\mathbb{P}(\xi_i=-1)=\frac{2}{3}=:q$, $\mathbb{P}(\xi_i=+1)=\frac{1}{3}=:p$.

Let $S_0=0$; $S_n=\xi_1+\ldots+\xi_n$. (S_n is the symple asymmetric random walk, not trapped).

Then $M_n := \left(\frac{q}{p}\right)^{S_n} = 2^{S_n}$ is a martingale adopted to the natural filtration $\mathcal{F}_n := \sigma(\xi_1, \ldots, \xi_n)$:

- Clearly M_n is bounded for every n (takes only finitely many values) so $E|M_n| < \infty$
- adopted by construction
- $\mathbb{E}(M_{n+1}|\mathcal{F}_n) = \mathbb{E}\left(\left(\frac{q}{p}\right)^{S_n+\xi_{n+1}}\Big|\mathcal{F}_n\right) = \mathbb{E}\left(M_n\left(\frac{q}{p}\right)^{\xi_{n+1}}\Big|\mathcal{F}_n\right) \underset{M_n \in \mathcal{F}_n}{=} M_n\,\mathbb{E}\left(\left(\frac{q}{p}\right)^{\xi_{n+1}}\Big|\mathcal{F}_n\right) \underset{\text{of } \mathcal{F}_n}{\overset{\xi_{n+1} \text{ is independent}}{=}} M_n\,\mathbb{E}\left(\left(\frac{q}{p}\right)^{\xi_{n+1}}\right) =$

$$= M_n\left[q\left(\frac{q}{p}\right)^{-1} + p\left(\frac{q}{p}\right)^{+1}\right] = M_n\left[\cancel{q}\frac{p}{\cancel{q}} + \cancel{p}\frac{q}{\cancel{p}}\right] = M_n \quad \checkmark$$

Let τ be the 1st hitting time of $\{-10, 10\}$ by S_n :

$\tau = \inf\{n \mid S_n \in \{-10, 10\}\}$. Then τ is a stopping time, so $M_{n\wedge\tau}$ is also a martingale, which is also bounded:

$M_{n\wedge\tau} = 2^{S_{n\wedge\tau}}$, so $2^{-10} \le M_{n\wedge\tau} < 2^{10}$. The martingale convergence thm says that $M_{n\wedge\tau}$ is convergent, so $\mathbb{P}(\tau<\infty)=1$.

Let $A = \{S_\tau = -10\}$; $B = \{S_\tau = +10\}$. We saw that $\boxed{\mathbb{P}(A)+\mathbb{P}(B)=1}$,

so ~~$\mathbb{E}S_\tau = -10\mathbb{P}(A)+10\mathbb{P}(B)$~~. $\mathbb{E}M_\tau = \mathbb{P}(A)2^{-10} + \mathbb{P}(B)2^{10}$.

The optional stopping thm says that $\mathbb{E}M_\tau = EM_0 = 1$, so

~~$\begin{cases}\mathbb{P}(A)+\mathbb{P}(B)=1 \\ 2^{-10}\mathbb{P}(A)+2^{10}\mathbb{P}(B)=1\end{cases}$ unique solution $\mathbb{P}(A)=\frac{2^{10}-1}{2^{10}-2^{-10}}=\frac{2^{20}-2^{10}}{2^{20}-1}$; $\mathbb{P}(B)=\frac{1-2^{-10}}{2^{10}-2^{-10}}=\frac{2^{10}-1}{2^{20}-1}$~~

5 $\boxed{\text{continued}}$

So
$$\begin{cases} \mathbb{P}(A)+\mathbb{P}(B)=1 \\ 2^{-10}\mathbb{P}(A)+2^{10}\mathbb{P}(B)=1 \end{cases} \xrightarrow{\times 2^{-10}} \begin{array}{l} 2^{-10}\mathbb{P}(A)+2^{-10}\mathbb{P}(B)=2^{-10} \\ 2^{-10}\mathbb{P}(A)+2^{10}\mathbb{P}(B)=1 \end{array}$$

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substract $\mathbb{P}(B)\left[2^{-10}-2^{10}\right]=2^{-10}-1$

$$\Rightarrow \underline{\underline{\mathbb{P}(B)}}=\frac{1-2^{-10}}{2^{10}-2^{-10}}=\frac{2^{10}-1}{2^{20}-1}=\frac{\cancel{2^{10}-1}}{\cancel{(2^{10}-1)}(2^{10}+1)}=\underline{\underline{\frac{1}{1025}}}$$