Probability 1
CEU Budapest, fall semester 2017
Imre Péter Tóth
Homework sheet 1 – solutions

1. Define a σ-algebra as follows:

   Definition 1 *For a nonempty set Ω, a family $\mathcal{F}$ of subsets of ω (i.e. $\mathcal{F} \subset 2^\Omega$, where $2^\Omega := \{A : A \subset \Omega\}$ is the power set of Ω) is called a σ-algebra over Ω if*

   - $\emptyset \in \mathcal{F}$
   - *if $A \in \mathcal{F}$, then $A^C := \Omega \setminus A \in \mathcal{F}$ (that is, $\mathcal{F}$ is closed under complement taking)*
   - *if $A_1, A_2, \dots \in \mathcal{F}$, then $(\cup_{i=1}^\infty A_i) \in \mathcal{F}$ (that is, $\mathcal{F}$ is closed under countable union).*

   Show from this definition that a σ-algebra is closed under countable intersection, and under finite union and intersection.

2. *Continuity of the measure*

   (a) Prove the following:

   Theorem 1 *(Continuity of the measure)*

   i. If $(\Omega, \mathcal{F}, \mu)$ is a measure space and $A_1, A_2, \dots$ is an increasing sequence of measurable sets (i.e. $A_i \in \mathcal{F}$ and $A_i \subset A_{i+1}$ for all i), then $\mu(\cup_{i=1}^\infty A_i) = \lim_{i\to\infty} \mu(A_i)$ (and both sides of the equation make sense).

   ii. If $(\Omega, \mathcal{F}, \mu)$ is a measure space, $A_1, A_2, \dots$ is a decreasing sequence of measurable sets (i.e. $A_i \in \mathcal{F}$ and $A_i \supset A_{i+1}$ for all i) and $\mu(A_1) < \infty$, then $\mu(\cap_{i=1}^\infty A_i) = \lim_{i\to\infty} \mu(A_i)$ (and both sides of the equation make sense).

   (b) Show that in the second statement the condition $\mu(A_1) < \infty$ is needed, by constructing a counterexample for the statement when this condition does not hold.

3. **(homework)**

   (a) We toss a biased coin, on which the probability of heads is some $0 \le p \le 1$. Define the random variable ξ as the indicator function of tossing heads, that is

   $$\xi := \begin{cases} 0, & \text{if tails} \\ 1, & \text{if heads} \end{cases} .$$

   i. Describe the distribution of ξ (called the Bernoulli distribution with parameter p) in the "classical" way, listing possible values and their probabilities,

   ii. and also by describing the distribution as a measure on $\mathbb{R}$, giving the weight $\mathbb{P}(\xi \in B)$ of every (Borel) subset B of $\mathbb{R}$.

   iii. Calculate the expectation of ξ.

   (b) We toss the previous biased coin n times, and denote by X the *number of heads* tossed.

   i. Describe the distribution of X (called the Binomial distribution with parameters (n, p)) by listing possible values and their probabilities.

   ii. Calculate the expectation of X by the old "probability 1" definition, using its distribution,

   iii. and also by noticing that $X = \xi_1 + \xi_2 + \dots + \xi_n$, where ξ_i is the indicator of the i-th toss being heads, and using linearity of the expectation.

Solution:

(a) i. The possible values are 0 and 1, their probabilities are $\mathbb{P}(\xi = 0) = 1 - p$ and $\mathbb{P}(\xi = 1) = p$.

ii. $\mu(B) = \mathbb{P}(\xi \in B) = \begin{cases} 1, & \text{if } 0 \in B \text{ and } 1 \in B, \\ 1-p, & \text{if } 0 \in B \text{ but } 1 \notin B, \\ p, & \text{if } 1 \in B \text{ but } 0 \notin B, \\ 0, & \text{if } 0 \notin B \text{ and } 1 \notin B. \end{cases}$

iii. $\mathbb{E}\xi = 0 \cdot \mathbb{P}(\xi = 0) + 1 \cdot \mathbb{P}(\xi = 1) = 0 \cdot (1-p) + 1 \cdot p = p.$

(b) i. The possible values are $0, 1, 2, \dots, n$, their probabilities are

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \dots, n.$$

ii. If we denote the distribution of X by μ, then

$$\mathbb{E}X = \int_{\mathbb{R}} x \, \mathrm{d}\mu(x) = \sum_{k=0}^{n} k \cdot \mu(\{k\}) = \sum_{k=0}^{n} k \cdot \mathbb{P}(X = k) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}.$$

To calculate this sum, one of the many ways is to consider the two-variable function

$$f(u, v) := \sum_{k=0}^{n} k \binom{n}{k} u^k v^{n-k}.$$

Then what we want to know is $\mathbb{E}X = f(p, 1-p)$, but of course we are even more happy if we can calculate $f(u, v)$ for every (u, v). Now we notice that

$$f(u, v) = u \frac{\partial}{\partial u} g(u, v) \text{ where } g(u, v) = \sum_{k=0}^{n} \binom{n}{k} u^k v^{n-k}.$$

This is now easy: by the binomial theorem $g(u, v) = (u + v)^n$, so

$$f(u, v) = u \frac{\partial}{\partial u} (u + v)^n = nu(u + v)^{n-1},$$

and

$$\mathbb{E}X = f(p, 1-p) = np(p + 1 - p)^n = np.$$

iii. This is much easier:

$$\mathbb{E}X = \mathbb{E}\Big(\sum_{i=1}^{n} \xi_i\Big) = \sum_{i=1}^{n} \mathbb{E}\xi_i = \sum_{i=1}^{n} p = np.$$

4. **(homework)** *Usefulness of the linearity of the expectation.* A building has 10 floors, not including the ground floor. On the ground floor, 10 people get into the elevator, and every one of them chooses a destination at random, uniformly out of the 10 floors, independently of the others. Let X denote the number of floors *on which the elevator stops* – i.e. the number of floors that were chosen by at least one person. Calculate the expectation of X. *(Hint: First notice that the distribution of X is hard to calculate. Find a way to calculate the expectation without that. Help: What is the probability that the elevator stops on the first floor?)*

Solution: For $i = 1, 2, \dots, 10$ let $Y_i = 1$ if the elevator stops on floor i, and let $Y_i = 0$ if not. So $X = Y_1 + \dots + Y_{10}$, which means that $\mathbb{E}X = \mathbb{E}Y_1 + \dots + \mathbb{E}Y_{10}$ (although the Y_i are not at all independent). These Y_i are just Bernoulli distributed with the same parameter

$$p = \mathbb{P}(Y_1 = 1) = \mathbb{P}(\text{stops on first floor}).$$

The probability of nobody going to the first floor is $\left(\frac{9}{10}\right)^{10}$, because the 10 people choose independently, and each of them pick another floor with probability $\frac{9}{10}$. So

$$p = 1 - \left(\frac{9}{10}\right)^{10},$$

and of course $\mathbb{E}Y_i = p$ for every i. This gives

$$\mathbb{E}X = 10p = 10\left[1 - \left(\frac{9}{10}\right)^{10}\right] \approx 6.51$$

5. **(homework)** We take a huge bag. 1 minute before midnight we put 10 balls (numbered $1 \dots 10$) into the bag. Then we draw a ball from the bag at random, and throw it away. $\frac{1}{2}$ minute before midnight we put another 10 balls (numbered $11 \dots 20$) into the bag. Then we draw a ball from the bag at random, and throw it away. $\frac{1}{4}$ minute before midnight we put another 10 balls (numbered $21 \dots 30$) into the bag. Then we draw a ball from the bag at random, and throw it away. And so on, infinitely many times: $\frac{1}{2^n}$ minute before midnight we put 10 balls (numbered $(10n+1)\dots(10n+10)$) into the bag. Then we draw a ball from the bag at random, and throw it away.

   a.) What is the probability that ball number 1 will be in the bag at midnight? *(Hint: we will see later that* $\lim_{N\to\infty} \prod_{n=0}^{N} \left(1 - \frac{1}{9n+10}\right) = 0$.*)*

   b.) What is the probability that ball number 11 will be in the bag at midnight?

   c.) Show that, at midnight, with probability 1, the bag will be empty. (What?!?!)

   Solution:

   a.) At each step we add 10 balls and take away 1, so after n steps there are $9n$ balls in the bag, and 10 more when we do the next draw. So, in step $n+1$ the chance of throwing away ball 1 is $\frac{1}{9n+10}$ – provided that it hasn't been thrown away before. The draws are independent, so

   $$\mathbb{P}(\text{ball 1 survives } N \text{ steps}) = \prod_{n=0}^{N-1} \left(1 - \frac{1}{9n+10}\right).$$

   To be in the bag at midnight, it needs to survive all the infinitely many steps, which has probability $\lim_{N\to\infty} \prod_{n=0}^{N} \left(1 - \frac{1}{9n+10}\right)$, which is 0 by the hint.
   Of course, balls 2, 3, ..., 10 have the same chance 0 of surviving.

   b.) Similarly to the previous case, the chance of surviving is $\lim_{N\to\infty} \prod_{n=1}^{N} \left(1 - \frac{1}{9n+10}\right)$, which differs from the previous only with the factor corresponding to $n = 0$, which is $1 - \frac{1}{10} \neq 0$. So, since the other limit is 0, this is also 0.
   Of course, balls 12, 13, ..., 20 have the same chance 0 of surviving.

   c.) With the same argument, each and every ball has 0 probability if surviving. Let A_i be the event that ball i survives, and let B be the event that there is at least 1 surviving ball (so the bag is not empty). So $B = \bigcup_{i=1}^{\infty} A_i$, and the σ-subadditivity of the probability implies that

   $$\mathbb{P}(B) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i) = \sum_{i=1}^{\infty} 0 = 0.$$

   This seemingly contradicts the fact that the number of balls is growing in each step, so after infinitely many steps there should be infinitely many balls in the bag. The contradiction comes from the problem not being formulated precisely. To see that there are two different questions here, let $A_{i,n}$ be the event that ball number i is in the bag after n steps. We have shown that

$$\mathbb{P}(B) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{\infty}\bigcap_{n} A_{i.n}\right) = 0,$$

which does not contradict the fact that

$$\lim_{n\to\infty} \sharp\{i \,|\, A_{i,n} \text{ holds}\} = \infty.$$

This is deeply related to the non-interchangeability of limit and integral – discussed in class.