Probability 1
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Imre Péter Tóth
Homework sheet 2 – solutions

2.1 *The Fatou lemma* is the following

Theorem 1 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f_1, f_2, \dots$ a sequence of measureabale functions $f_n : \Omega \to \mathbb{R}$, which are nonneagtive, e.g. $f_n(x) \geq 0$ for every $n = 1, 2, \dots$ and every $x \in \Omega$. Then*

$$\int_\Omega \liminf_{n\to\infty} f_n(x)\,\mathrm{d}\mu(x) \leq \liminf_{n\to\infty} \int_\Omega f_n(x)\,\mathrm{d}\mu(x)$$

(and both sides make sense).

Show that the inequality in the opposite direction is in general false, by choosing $\Omega = \mathbb{R}$, μ as the Lebesgue measure on $\mathbb{R}$, and constructing a sequence of nonnegative $f_n : \mathbb{R} \to \mathbb{R}$ for which $f_n(x) \xrightarrow{n\to\infty} 0$ for every $x \in \mathbb{R}$, but $\int_\mathbb{R} f_n(x)\,\mathrm{d}x \geq 1$ for all n.

2.2 **(homework)** *Exchangeability of integral and limit.* Consider the sequences of functions $f_n : [0,1] \to \mathbb{R}$ and $g_n : [0,1] \to \mathbb{R}$ concerning their pointwise limits and the limits of their integrals. Do there exist integrable functions $f : [0,1] \to \mathbb{R}$ and $g : [0,1] \to \mathbb{R}$, such that $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ for Lebesgue almost every $x \in [0,1]$? What is $\lim\limits_{n\to\infty} \left(\int\limits_0^1 f_n(x)dx \right)$ and $\lim\limits_{n\to\infty} \left(\int\limits_0^1 g_n(x)dx \right)$? Are the conditions of the dominated and monotone convergence theorems and the Fatou lemma satisfied? If yes, what do these theorems ensure about these specific examples? (For the Fatou lemma, see the lecture notes or Exercise 1.)

(a)

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \leq x < 1/n, \\ 2n - n^2 x & \text{if } 1/n \leq x \leq 2/n, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Write n as $n = 2^k + l$, where $k = 0, 1, 2 \dots$ and $l = 0, 1, \dots, 2^k - 1$ (this can be done in a unique way for every n). Now let

$$g_n(x) = \begin{cases} 1 & \text{if } \frac{l}{2^k} \leq x < \frac{l+1}{2^k}, \\ 0 & \text{otherwise.} \end{cases}$$

Solution: Please **draw the graph of these functions!** You will see that

(a) $f_n(x) \to f(x) := 0$ for every $x \in [0,1]$, but $\int\limits_0^1 f_n(x)dx = 1$ for every n (except for $n = 1$), so

$$\lim_{n\to\infty} \left(\int\limits_0^1 f_n(x)dx \right) = 1 \neq 0 = \int\limits_0^1 f(x)dx.$$

The convergence is not monotone, so the monotone convergence theorem says nothing. A natural choice for the common dominating function would be $G(x) := \frac{1}{x}$ (see the graphs you have drawn), but this is not integrable, because $\int\limits_0^1 \frac{1}{x}dx = \infty$. Indeed, there is no integrable dominating function, so the dominated convergence theorem says nothing as well. However, the functions are non-neagtive, so the conditions of the Fatou lemma hold. Of course, the statement also holds:

$$\int\limits_0^1 \liminf_{n\to\infty} f_n(x)dx = 0 \le 1 = \liminf_{n\to\infty} \int\limits_0^1 f_n(x)dx.$$

(b) For every fixed $x \in [0,1]$ as n grows, $g_n(x)$ will be 0 most of the time, but it will also be 1 once in a while (infinitely many times). So $g_n(x)$ is not convergent (as $n \to \infty$) for any x: no limiting g exists. Thus the monotone and dominated convergence theorems say nothing (there is no $\int_0^1 g(x)dx$ to converge to). However, the conditions of the Fatou lemma hold. Of course, the statement also holds: $\liminf_{n\to\infty} g_n(x) = 0$ for every x, while $\int\limits_0^1 g_n(x)dx = \frac{1}{2^k}$ for $2^k \le n \le 2^{k+1}$, so $\int\limits_0^1 g_n(x)dx \xrightarrow{n\to\infty} 0$. The statement of the lemma now reads

$$\int\limits_0^1 \liminf_{n\to\infty} g_n(x)dx = 0 \le 0 = \liminf_{n\to\infty} \int\limits_0^1 g_n(x)dx.$$

2.3 **(homework)** *Exchangeability of integrals.* Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x,\ 0 < y \text{ and } 0 \le x - y \le 1, \\ -1 & \text{if} \quad 0 < x,\ 0 < y \text{ and } 0 < y - x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $\int\limits_{-\infty}^{+\infty} \left(\int\limits_{-\infty}^{+\infty} f(x,y)dx \right) dy$ and $\int\limits_{-\infty}^{+\infty} \left(\int\limits_{-\infty}^{+\infty} f(x,y)dy \right) dx$. What's the situation with the Fubini theorem?

Solution: Sketching the function one easily sees that

$$\int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}x = \begin{cases} 1-y, \text{ if } 0 < y < 1 \\ 0, \text{ if not} \end{cases},$$

so $\int\limits_{-\infty}^{+\infty} \left(\int\limits_{-\infty}^{+\infty} f(x,y)dx \right) dy = \int\limits_0^1 (1-y)dy = \frac{1}{2}$. Similarly

$$\int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}y = \begin{cases} -1+x, \text{ if } 0 < x < 1 \\ 0, \text{ if not} \end{cases},$$

so $\int\limits_{-\infty}^{+\infty} \left(\int\limits_{-\infty}^{+\infty} f(x,y)dy \right) dx = \int\limits_0^1 (x-1)dx = -\frac{1}{2}$. The two double integrals are not equal, but this does not contradict the Fubini theorem, because f is not integrable (w.r.t. Lebesgue measure on $\mathbb{R}^2$). Indeed, $\iint_{\mathbb{R}^2} |f| = \infty$.

2.4 The characteristic function of a random variable X is the function $\Psi : \mathbb{R} \to \mathbb{C}$ defined as $\Psi(t) := \mathbb{E}e^{itX}$, which, of course, depends on the distribution of X only. Calculate the characteristic function of

(a) The Bernoulli distribution $B(p)$ (see Homework sheet 1)

(b) The "pessimistic geometric distribution with parameter p" – that is, the distribution μ on $\{0, 1, 2 \dots\}$ with weights $\mu(\{k\}) = (1-p)p^k$ $(k = 0, 1, 2 \dots)$.

(c) The "optimistic geometric distribution with parameter p" – that is, the distribution ν on $\{1, 2, 3, \dots\}$ with weights $\nu(\{k\}) = (1-p)p^{k-1}$ $(k = 1, 2 \dots)$.

(d) **(homework)** The Poisson distribution with parameter λ – that is, the distribution η on $\{0, 1, 2 \dots\}$ with weights $\eta(\{k\}) = e^{-\lambda}\frac{\lambda^k}{k!}$ $(k = 0, 1, 2 \dots)$.
Solution:

$$\psi_{Poi(\lambda)}(t) = \sum_{k=0}^{\infty} e^{itk} e^{-\lambda}\frac{\lambda^k}{k!} = \sum_{k=0}^{\infty} e^{itk}\eta(\{k\}) = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda e^{it})^k}{k!} = e^{-\lambda}e^{\lambda e^{it}} = e^{\lambda(e^{it}-1)}.$$

(e) **(homework)** The exponential distribution with parameter λ – that is, the distribution on $\mathbb{R}$ with density (w.r.t. Lebesgue measure)

$$f_\lambda(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{if not} \end{cases} .$$

Solution:

$$\phi_{Exp(\lambda)}(t) = \int_{\mathbb{R}} e^{itx} f_\lambda(x)\,\mathrm{dLeb}(x) = \int_0^\infty e^{itx}\lambda e^{-\lambda x}\,\mathrm{d}x = \lambda\left[\frac{e^{(it-\lambda)x}}{it-\lambda}\right]_0^\infty = \frac{\lambda}{\lambda - it}.$$

2.5 Calculate the characteristic function of the normal distribution $\mathcal{N}(m, \sigma^2)$. (Remember the definition from the old times: $\mathcal{N}(m, \sigma^2)$ is the distribution on $\mathbb{R}$ with density (w.r.t. Lebesgue measure)

$$f_{m,\sigma^2}(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-m)^2}{2\sigma^2}}.$$

You can save yourself some paperwork if you only do the calculation for $\mathcal{N}(0, 1)$ and reduce the general case to this using the relation between different normal distributions. You can and should use the fact that

$$\int_{-\infty}^{\infty} f_{m,\sigma^2}(x)\,\mathrm{d}x = 1$$

for every m and σ.

2.6 *Dominated convergence and continuous differentiability of the characteristic function.*
The Lebesgue dominated convergence theorem is the following

Theorem 2 (dominated convergence) *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f_1, f_2, \dots$ measurable real valued functions on Ω which converge to the limit function pointwise, μ-almost everywehere. (That is, $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in \Omega$, except possibly for a set of x-es with μ-measure zero.) Assume furthermore that the f_n admit a common* integrable dominating function*: there exists a $g : \Omega \to \mathbb{R}$ such that $|f_n(x)| \le g(x)$ for every $x \in \Omega$ and $n \in \mathbb{N}$, and $\int_\Omega g\,\mathrm{d}\mu < \infty$. Then (all the f_n and also f are integrable and)*

$$\lim_{n\to\infty}\int_\Omega f_n\,\mathrm{d}\mu = \int_\Omega f\,\mathrm{d}\mu.$$

Use this theorem to prove the following

Theorem 3 (differentiability of the characteristic function) *Let X be a real valued random variable, $\psi(t) = \mathbb{E}(e^{itX})$ its characteristic function and $n \in \mathbb{N}$. If the n-th moment of X exists and is finite (i.e. $\mathbb{E}(|X|^n) < \infty$), then ψ is n times continuously differentiable and*

$$\psi^{(k)}(0) = i^k \mathbb{E}(X^k), \quad k = 0, 1, 2, \ldots, n.$$