Probability 1
CEU Budapest, fall semester 2018
Imre Péter Tóth
Homework sheet 3 – solutions

3.1 The characteristic function of a random variable X is the function $\Psi : \mathbb{R} \to \mathbb{C}$ defined as $\Psi(t) := \mathbb{E}e^{itX}$, which, of course, depends on the distribution of X only. Calculate the characteristic function of

(a) The Bernoulli distribution $B(p)$

(b) The "pessimistic geometric distribution with parameter p" – that is, the distribution μ on $\{0, 1, 2 \dots\}$ with weights $\mu(\{k\}) = (1 - p)p^k$ $(k = 0, 1, 2 \dots)$.

(c) The "optimistic geometric distribution with parameter p" – that is, the distribution ν on $\{1, 2, 3, \dots\}$ with weights $\nu(\{k\}) = (1 - p)p^{k-1}$ $(k = 1, 2 \dots)$.

(d) The Poisson distribution with parameter λ – that is, the distribution η on $\{0, 1, 2 \dots\}$ with weights $\eta(\{k\}) = e^{-\lambda}\frac{\lambda^k}{k!}$ $(k = 0, 1, 2 \dots)$.

(e) The exponential distribution with parameter λ – that is, the distribution on $\mathbb{R}$ with density (w.r.t. Lebesgue measure)

$$f_\lambda(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{if not} \end{cases} .$$

3.2 Calculate the characteristic function of the normal distribution $\mathcal{N}(m, \sigma^2)$. (Remember the definition from the old times: $\mathcal{N}(m, \sigma^2)$ is the distribution on $\mathbb{R}$ with density (w.r.t. Lebesgue measure)

$$f_{m,\sigma^2}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}.$$

You can save yourself some paperwork if you only do the calculation for $\mathcal{N}(0, 1)$ and reduce the general case to this using the relation between different normal distributions. You can and should use the fact that

$$\int_{-\infty}^{\infty} f_{m,\sigma^2}(x)\,\mathrm{d}x = 1$$

for every m and σ.

3.3 *Dominated convergence and continuous differentiability of the characteristic function.*
The Lebesgue dominated convergence theorem is the following

Theorem 1 (dominated convergence) *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f_1, f_2, \dots$ measurable real valued functions on Ω which converge to the limit function pointwise, μ-almost everywehere. (That is, $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in \Omega$, except possibly for a set of x-es with μ-measure zero.) Assume furthermore that the f_n admit a common* integrable dominating function*: there exists a $g : \Omega \to \mathbb{R}$ such that $|f_n(x)| \le g(x)$ for every $x \in \Omega$ and $n \in \mathbb{N}$, and $\int_\Omega g\,\mathrm{d}\mu < \infty$. Then (all the f_n and also f are integrable and)*

$$\lim_{n\to\infty} \int_\Omega f_n\,\mathrm{d}\mu = \int_\Omega f\,\mathrm{d}\mu.$$

Use this theorem to prove the following

Theorem 2 (differentiability of the characteristic function) *Let X be a real valued random variable, $\psi(t) = \mathbb{E}(e^{itX})$ its characteristic function and $n \in \mathbb{N}$. If the n-th moment of X exists and is finite (i.e. $\mathbb{E}(|X|^n) < \infty$), then ψ is n times continuously differentiable and*

$$\psi^{(k)}(0) = i^k \mathbb{E}(X^k), \quad k = 0, 1, 2, \ldots, n.$$

Write the proof in detail for $n = 1$. Don't forget about proving *continuous* differentiability – meaning that you also have to check that the derivative is continuous.

3.4 For real numbers $a_1, a_2, a_3, \ldots$ define the infinite product $\prod_{k=1}^{\infty} a_k$ as

$$\prod_{k=1}^{\infty} a_k := \lim_{n\to\infty} \prod_{k=1}^{n} a_k,$$

whenever this limit exists.

Let $p_1, p_2, p_3, \ldots$ satisfy $0 \le p_k < 1$ for all k. Show that $\prod_{k=1}^{\infty}(1 - p_k) > 0$ if and only if $\sum_{k=1}^{\infty} p_k < \infty$.

(Hint: estimate the logarithm of $(1 - p)$ with p.)

3.5 Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with $\mathbb{E}X_1 = 0$ and $\mathbb{E}(X_1^4) < \infty$ and set $S_n = X_1 + \cdots + X_n$. Show that there is a $C < \infty$ such that $\mathbb{E}(S_n^4) \le Cn^2$.

3.6 **(homework)** Let $X_1, X_2, \ldots$ be independent random variables such that

$$\mathbb{P}(X_n = n^2 - 1) = \frac{1}{n^2}, \quad \mathbb{P}(X_n = -1) = 1 - \frac{1}{n^2}.$$

Show that $\mathbb{E}X_n = 0$ for every n, but

$$\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = -1$$

almost surely.

Solution:

$$\mathbb{E}X_n = -1\mathbb{P}(X_n = -1) + (n^2 - 1)\mathbb{P}(X_n = n^2 - 1) = -1 + \frac{1}{n^2} + \frac{n^2 - 1}{n^2} = 0.$$

Now define the events $A_n := \{X_n \neq -1\}$. Then $\sum_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, so the first Borel-Cantelli lemma says that with probability 1 only finitely many A_n occur. In particular, $X_n \to -1$ almost surely. Then of course $\frac{X_1+\cdots+X_n}{n} \to -1$ almost surely as well.

3.7 Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables. Prove that the following two statements are equivalent:

(i) $\mathbb{E}|X_i| < \infty$.

(ii) $\mathbb{P}(|X_n| > n$ for infinitely many n-s$) = 0$.

Hint: If Y is nonnegative integer valued, then $\mathbb{E}Y = \sum_{k=0}^{\infty} k\mathbb{P}(Y = k) = \sum_{n=1}^{\infty} \mathbb{P}(Y \ge n)$. (Why?)

3.8 Prove that for *any* sequence $X_1, X_2, \dots$ of random variables (real valued, defined on the same probability space) there exists a sequence $c_1, c_2, \dots$ of numbers such that

$$\frac{X_n}{c_n} \to 0 \text{ almost surely.}$$

3.9 Let the random variables $X_1, X_2, \dots, X_n, \dots$ and X be defined on the same probability space. Prove that the following two statements are equivalent:

(i) $X_n \to X$ in probability as $n \to \infty$.

(ii) From every subsequence $\{n_k\}_{k=1}^\infty$ a sub-subsequence $\{n_{k_j}\}_{j=1}^\infty$ can be chosen such that $X_{n_{k_j}} \to X$ almost surely as $j \to \infty$.

3.10 Let $X_1, X_2, \dots$ be independent such that X_n has *Bernoulli*(p_n) distribution. Determine what property the sequence p_n has to satisfy so that

(a) $X_n \to 0$ in probability as $n \to \infty$

(b) $X_n \to 0$ almost surely as $n \to \infty$.

3.11 Let $X_1, X_2, \dots$ be independent random variables. Show that $\mathbb{P}(\sup_n X_n < \infty) = 1$ if and only if there is some $A \in \mathbb{R}$ for which $\sum_{n=1}^\infty \mathbb{P}(X_n > A) < \infty$.

3.12 Let $X_1, X_2, \dots$ be independent exponentially distributed random variables such that X_n has parameter λ_n. Let $S_n := \sum_{i=1}^n X_i$. Show that if $\sum_{n=1}^\infty \frac{1}{\lambda_n} = \infty$, then $S_n \to \infty$ almost surely, but if $\sum_{n=1}^\infty \frac{1}{\lambda_n} < \infty$, then $S_n \to S$ almost surely, where S is some random variable which is almost surely finite. *(Hint: the second part is easy. For the first part, a possible solution is to let x_i be such that $\mathbb{P}(X_i \geq x_i) = \frac{1}{2}$, $Y_i := x_i \mathbf{1}_{\{X_i \geq x_i\}}$, $Z_i := x_i - Y_i$ and use that $S_n \geq \sum_{i=1}^n Y_i$.)*

3.13 **(homework)** Let $X_1, X_2, \dots$ be i.i.d. random variables with distribution *Bernoulli*(p) for some $p \in (0; 1)$ but $p \neq \frac{1}{2}$. Let $Y := \sum_{n=1}^\infty 2^{-n} X_n$. (The sum is absolutely convergent.) Show that the distribution ν of Y is continuous (meaning that the distribution function is continuous, which is the same as $\nu(\{x\}) = 0$ for any $x \in \mathbb{R}$), but singular w.r.t. Lebesgue measure (meaning that there is a set $A \subset \mathbb{R}$ such that $Leb(A) = 0$ and $\nu(\mathbb{R} \setminus A) = 0$).

(Hint: Think of these random numbers as sequences of 0s and 1s in binary form. What will be the proportion of 0s and 1s?)

Solution: Clearly $0 \leq Y \leq 1$ always, so $\nu(\mathbb{R} \setminus [0, 1]) = \mathbb{P}(Y \notin [0, 1]) = 0$. If $y \in [0, 1]$, then there are either 1 or 2 sequences of bits $x_n \in \{0, 1\}$ which produce $\sum_{n=1}^\infty 2^{-n} x_n = y$. (Indeed, the binary expansion is unique for most numbers, and only the numbers of the form $\frac{l}{2^k}$ have two expansions, e.g. $0.10100000\dot{0} = 0.10011111\dot{1}$.) But the probability of each sequence is zero: for a fixed sequence $x_1, x_2, \dots$

$$\mathbb{P}(X_i = x_i \text{ for every } i) \leq \mathbb{P}(X_1 = x_1, \dots X_n = x_n) \leq (\max\{p, 1-p\})^n$$

for every n, so $\mathbb{P}(X_i = x_i \text{ for every } i) = 0$. This means that $\mathbb{P}(Y = y) = 0$ for every y, so ν is continuous.

Now let $A = A_p$ be the set of those numbers in $[0, 1]$ whose binary expansion is such that the proportion of 1s converges to p:

$$A := A_p := \left\{ y \in [0, 1] \,\middle|\, y = \sum_{n=1}^\infty \frac{a_n}{2^n} \text{ with } a_n \in \{0, 1\} \text{ and } \frac{a_1 + \cdots + a_n}{n} \to p \right\}.$$

The strong law of large numbers says that $\nu(A) = \mathbb{P}(Y \in A) = 1$, because $\mathbb{P}(\frac{X_1+\cdots+X_n}{n} \to 1) = 1$. So $\nu(\mathbb{R} \setminus A) = 0$.

If, instead of $p \neq \frac{1}{2}$ we took $p = \frac{1}{2}$, then Y would be uniform on $[0,1]$, so its distribution ν would be Lebesgue measure (restricted to $[0,1]$). So again, the strong law of large numbers says that $Leb_{[0,1]}\left(A_{\frac{1}{2}}\right) = 1$. Since $A = A_p$ and $A_{\frac{1}{2}}$ are disjoint, $Leb(A) = 0$, so ν is indeed singular w.r.t Lebesgue measure.

3.14 Let the random variables $X_1, X_2, \ldots, X_n, \ldots$ and X be defined on the same probability space and suppose that $X_n \to X$ in probability as $n \to \infty$.

(a) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $Y_n = f(X_n)$ and $Y = f(X)$, show that $Y_n \to Y$ in probability as $n \to \infty$.

(b) Show that if the X_n are almost surely uniformly bounded [that is: there exists a constant $M < \infty$ such that $\mathbb{P}(\forall n \in \mathbb{N}\, |X_n| \leq M) = 1$], then $\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X$.

(c) Show, through an example, that for the previous statement, the condition of boundedness is needed.

3.15 **(homework)** Let the random variables $X_1, X_2, \ldots, Y_1, Y_2, \ldots, X$ and Y be defined on the same probability space and assume that $X_n \to X$ and $Y_n \to Y$ in probability. Show that

(a) $X_nY_n \to XY$ in probability.

(b) If almost surely $Y_n \neq 0$ and $Y \neq 0$, then $X_n/Y_n \to X/Y$ in probability.

Solution:

(a)

$$\begin{aligned} |X_nY_n - XY| &= |X_n(Y_n - Y) + (X_n - X)Y| \leq |X_n||Y_n - Y| + |X_n - X||Y| \\ &\leq (|X| + |X_n - X|)|Y_n - Y| + |X_n - X||Y|. \end{aligned} \tag{1}$$

Let $\delta > 0$ and $\varepsilon > 0$. We will show that if n is big enough, then $\mathbb{P}(|X_nY_n - XY| \geq \delta) \leq \varepsilon$. For this purpose,

i. let M be so big that $\mathbb{P}(|Y| \geq M) \leq \frac{\varepsilon}{4}$ and $\mathbb{P}(|X| \geq M) \leq \frac{\varepsilon}{4}$,

ii. let n be so big that
- $\mathbb{P}\left(|X_n - X| \geq \frac{\delta}{2M}\right) \leq \frac{\varepsilon}{4}$
- $\mathbb{P}\left(|Y_n - Y| \geq \frac{\delta}{2\left(M + \frac{\delta}{2M}\right)}\right) \leq \frac{\varepsilon}{4}$.

Then on some event A with probability at least $1 - 4\frac{\varepsilon}{4} = 1 - \varepsilon$ we have that

- $|Y| \leq M$
- $|X| \leq M$
- $|X - X_n| \leq \frac{\delta}{2M}$
- $|Y - Y_n| \leq \frac{\delta}{2\left(M + \frac{\delta}{2M}\right)}$.

Writing these back to (1), we get that on the set A

$$|X_nY_n - XY| \leq \left(M + \frac{\delta}{2M}\right) \frac{\delta}{2\left(M + \frac{\delta}{2M}\right)} + \frac{\delta}{2M}M = \delta.$$

(b) Because of the previous point, it is enough to show the statement for $X_n = X \equiv 1$. Then

$$\left|\frac{1}{Y_n} - \frac{1}{Y}\right| \leq \frac{|Y - Y_n|}{|Y_n||Y|}. \tag{2}$$

Let $\delta > 0$ and $\varepsilon > 0$. We will show that if n is big enough, then $\mathbb{P}\left(\left|\frac{1}{Y_n} - \frac{1}{Y}\right| \geq \delta\right) \leq \varepsilon$. For this purpose,

i. let $c > 0$ be so small that $\mathbb{P}(|Y| \leq c) \leq \frac{\varepsilon}{2}$,

ii. let n be so big that $\mathbb{P}\left(|Y_n - Y| \geq \max\left\{\frac{c}{2}, \delta\frac{c^2}{2}\right\}\right) \leq \frac{\varepsilon}{2}$.

Then on some event A with probability at least $1 - 2\frac{\varepsilon}{2} = 1 - \varepsilon$ we have that

- $|Y| \geq c$
- $|Y - Y_n| \leq \frac{c}{2}$, so $|Y_n| \geq \frac{c}{2}$
- $|Y - Y_n| \leq \delta\frac{c^2}{2}$.

Writing these back to (2), we get that on the set A

$$\left|\frac{1}{Y_n} - \frac{1}{Y}\right| \leq \frac{\delta\frac{c^2}{2}}{\frac{c}{2}c} = \delta.$$

3.16 **(homework)** Let the random variables $X_1, X_2, \ldots, X_n, \ldots$ be defined on the same probability space and let $Y_n := \sup_{m \geq n} |X_m|$. Prove that the following two statements are equivalent:

(i) $X_n \to 0$ almost surely as $n \to \infty$.

(ii) $Y_n \to 0$ in probability as $n \to \infty$.

Solution: For any sequence of numbers a_n, if we set $b_n := \sup_{m \geq n} |a_m|$, then we get $b_n \to 0$ if and only if $a_n \to 0$. Moreover, b_n is automatically monotone decreasing. So the events $\{Y_n \to 0\}$ and $\{X_n \to 0\}$ are the same, so $X_n \to 0$ almost surely if and only if $Y_n \to 0$ almost surely. This of course implies that $Y_n \to 0$ in probability.

Now since Y_n is monotone decreasing, convergence to 0 in probability also implies convergence to 0 almost surely: if there were a set of positive measure where $Y_n \not\to 0$, then on some (possibly smaller) positive measure set Y_n would stay bigger than some $\varepsilon > 0$ for ever, which contradicts convergence in probability.