Probability 1
CEU Budapest, fall semester 2017
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Homework sheet 6 – solutions

6.1 Durrett [1], Exercise 5.2.13

6.2 **(homework)** Let X_n be a simple random walk on $\mathbb{Z}$ starting from $X_0 = 0$. (As before, this means that $X_n = \xi_1 + \xi_2 + \cdots + \xi_n$, where the ξ_i are i.i.d. with $\mathbb{P}(X_i = 1) = p = 1 - \mathbb{P}(X_i = -1)$, and $p \in [0, 1]$. (p need not be $\frac{1}{2}$, so the walk may be asymmetric.) Use the martingale convergence theorem to show that

a.) the walk reaches the set $\{-20, 30\}$ with probability 1.

b.) If $p \geq \frac{1}{2}$, then the walk reaches the point 30 with probability 1.

c.) If $p \leq \frac{1}{2}$, then the walk reaches the point -20 with probability 1.

Solution:

a.) Assume first that $p \geq \frac{1}{2}$, Then, as we have seen is class, X_n is a submartingale. Let $\tau := \inf\{n | X_n \in \{-20, 30\}\}$ be the first hitting time of the set $\{-20, 30\}$. This τ is a stopping time, so the stopped process $X_{n\wedge\tau}$ is also a submartingale. $X_{n\wedge\tau}$ is also bounded, so the martingale convergence theorem says that it is almost surely convergent. But, since it is integer valued, it can only be convergent if it is eventually constant – meaning no more jumps after a while. By construction, this can only happen if τ is reached, so $\mathbb{P}(\tau < \infty) = 1$.
Assume now that $p \leq \frac{1}{2}$. Now $Y_n := -X_n$ is a submartingale, and the same argument as above works with the same stopping time $\tau := \inf\{n | X_n \in \{-20, 30\}\} = \inf\{n | Y_n \in \{-30, 20\}\}$.

b.) If $p \geq \frac{1}{2}$, X_n is a submartingale. Now let $\tau := \inf\{n | X_n = 30\}$ be the first hitting time of the point 30. Again, τ is a stopping time, so the stopped process $X_{n\wedge\tau}$ is also a submartingale. This time $X_{n\wedge\tau}$ is only bounded from above, but the martingale convergence theorem still applies, so $X_{n\wedge\tau}$ is almost surely convergent, which means $\mathbb{P}(\tau < \infty) = 1$.

c.) If $p \leq \frac{1}{2}$, then $Y_n := -X_n$ is a submartingale, s if we set $\tau := \inf\{n | X_n = -20\} = \inf\{n | Y_n = 20\}$, then $Y_{n\wedge\tau}$ is a submartingale which is bounded from above, so the martingale convergence theorem applies, thus $X_{n\wedge\tau}$ is almost surely convergent, which means $\mathbb{P}(\tau < \infty) = 1$.

6.3 *(Pólya's urn)* In an urn there is initially (at time $n = 0$) a black and a white ball. At each time step $n = 1, 2, \dots$

- we draw a ball from the urn, uniformly at random,
- we look at its colour,
- we put it back, and we add another ball of the same colour.

(So we add exactly one ball in each step.) Let X_n be the number of white balls in the urn after n steps, and let $M_n = \frac{X_n}{n+2}$ be the proportion of white balls after n steps.

a.) Show that X_n is uniform on $\{1, 2, \dots, n + 1\}$. *(Hint: a possible solution is by induction.)*

b.) Show that M_n is almost surely convergent.

c.) What is the distribution of $M_\infty := \lim_{n\to\infty} M_n$?

6.4 **(homework)** In the (French style) Roulette, if you bet on "red", you lose your bet with probability $\frac{19}{37}$, and you win the amount of your bet with the remaining probability $\frac{18}{37}$. (E.g. if you bet on "red" with HUF 1 and you win, then you get your HUF 1 back, plus you get another HUF 1 as your winning.

You arrive at the casino with some money in your pocket, and keep betting on "red". At each spin, your bet may be anything between 0 and the amount of money you have. Let X_n be the amount of your money after n spins. Show that – no matter what your strategy is – X_n is convergent with probability 1.

Solution: Let $\xi_n = 1$ if the nth spin gives "red", and $\xi_n = -1$ if not. This is an i.i.d. sequence. The game is unfavourable, meaning $\mathbb{E}\xi_n < 0$, so the sum $S_n := \xi_1 + \cdots + \xi_n$ is a supermartingale. If your bet in the nth step is H_n, then your money at time n is the discrete stochastic integral $X_n := (H \bullet S)_n$. Now **let us assume** that you don't see the future, so H_n is predictable. *(This is not written in the exercise, but we have to assume it in the name of common sense. Without this assumption, the statement is false.)* Now since $H_n \geq 0$ by assumption, X_n is also a supermartingle. It is also non-negative by assumption, so the martingale convergence theorem says that it is almost surely convergent.

6.5 Alice and Bob keep tossing a possibly biased coin. Before each toss, they agree on a stake: Alice will give this sum to Bob if the coin turns "heads", and Bob will give the (same) sum to Alice if it turns "tails". The stake has to be a non-negative multiple of 1 penny, and they are not allowed to risk more money than what they have. If they agree on a stake which is 0, then the game ends. Show that sooner or later the game will end.

6.6 **(homework)** Harry is organizing a *pyramid scheme* in his family.
(See http://en.wikipedia.org/wiki/Pyramid_scheme) The participants are not too persistent: every participant keeps trying to recruit new participants until the first failure (i.e. until he is first rejected). The probability of such a failure is p at every recruit attempt, independently of the history of the scheme.

The first participant is Harry, he forms the 0-th generation alone. The first generation consists of those recruited (directly) by Harry. The second generation consists of those recruited (directly) by members of the first generation, and so on.

Let Z_k denote the size of the k-th generation ($k = 0, 1, 2, \ldots$), and let N denote the total number of participants in the scheme (meaning $N = \sum_{k=0}^{\infty} Z_k$).

0-*th question: What is the distribution of* Z_1 *(which is the same as the distribution of the number of participants recruited by any fixed member of the scheme)? This distribution has a name.*

Answer the questions below

I. for $p = \frac{2}{3}$,

II. for $p = \frac{1}{2}$,

III. for $p = \frac{1}{3}$:

a.) Let r be the probability that the scheme dies out (that is, one of the generations will already be empty). Is $r = 1$?

b.) What is the expectation of Z_n?

c.) What is the expectation of N?

d.) In case "not dying out" has positive probability, what is the growth rate of Z_n on this event?

Solution: 0-*th question:* Let $q = 1 - p$. Successfully recruiting k people means k successes and then 1 failure, so

$$\mathbb{P}(Z_1 = k) = q^k p, \quad k = 0, 1, 2, \dots.$$

So Z_1 has a "pessimistic geometric distribution" with parameter p. As a result, the expectation is $m = \mathbb{E}Z_1 = \frac{1}{p} - 1$.

From the description it follows that Z_n is a Galton-Watson branching process with $Z_0 = 1$.

I. If $p = \frac{2}{3}$, then $m = \frac{1}{p} - 1 = \frac{1}{2} < 1$, so the process is sub-critical. This implies that

   a.) $\mathbb{P}(\text{extinction}) = 1$.

   b.) $\mathbb{E}Z_n = m^n = \frac{1}{2^n}$.

   c.) $\mathbb{E}N = \sum_{n=0}^{\infty} \mathbb{E}Z_n = \sum_{n=0}^{\infty} m^n = \frac{1}{1-m} = 2$.

   d.) The question is not relevant: "not dying out" has zero probability.

II. If $p = \frac{1}{2}$, then $m = \frac{1}{p} - 1 = 1$, so the process is critical. This implies that

   a.) $\mathbb{P}(\text{extinction}) = 1$. (A critical process always dies out unless it is degenerate such that everybody has exactly 1 child.)

   b.) $\mathbb{E}Z_n = m^n = 1^n = 1$.

   c.) $\mathbb{E}N = \sum_{n=0}^{\infty} \mathbb{E}Z_n = \sum_{n=0}^{\infty} m^n = \sum_{n=0}^{\infty} 1 = \infty$.

   d.) The question is not relevant: "not dying out" has zero probability.

III. If $p = \frac{1}{3}$, then $m = \frac{1}{p} - 1 = 2$, so the process is super-critical. This implies that

   a.) $\mathbb{P}(\text{extinction}) < 1$.

   b.) $\mathbb{E}Z_n = m^n = 2^n$.

   c.) $\mathbb{E}N = \sum_{n=0}^{\infty} \mathbb{E}Z_n = \sum_{n=0}^{\infty} m^n = \sum_{n=0}^{\infty} 2^n = \infty$.

   d.) This time the question is relevant: "not dying out" has positive probability. We know that $\frac{Z_n}{m^n}$ is a non-negative martingale, so the martingale convergence theorem says that $W_n := \frac{Z_n}{m^n}$ converges to some W_∞. So $Z_n \sim W_\infty m^n = W_\infty 2^n$. We have not seen this, but it is true that $W_\infty > 0$ on the event $\{\text{no extinction}\}$ (which happens to have probability $\frac{1}{2}$).

References

[1] Durrett, R. *Probability: Theory and Examples.* **4th** edition, Cambridge University Press (2010)