**Probability 1**
**CEU Budapest, fall semester 2018**
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**Homework sheet 6 – solutions**

6.1 Consider the probability space $\Omega = \{a, b, c\}$ equipped with the uniform measure as $\mathbb{P}$ (so $\mathbb{P}(\{a\}) = \mathbb{P}(\{b\}) = \mathbb{P}(\{c\}) = \frac{1}{3}$). Let the random variable $X : \Omega \to \mathbb{R}$ be such that $X(a) = X(b) = 0$, $X(c) = 1$.

a.) Let $D_1$ be the partition $\{\{a\}, \{b, c\}\}$. Find the conditional expectation $\mathbb{E}(X|D_1)$ (which is the same as $\mathbb{E}(X|G_1)$, where the $\sigma$-algebra $G_1$ is $G_1 = \{\emptyset, \{a\}, \{b, c\}, \Omega\}$.)

b.) Let $D_2$ be the partition $\{\{a, b\}, \{c\}\}$. Find the conditional expectation $\mathbb{E}(X|D_2)$ (which is the same as $\mathbb{E}(X|G_2)$, where the $\sigma$-algebra $G_2$ is $G_2 = \{\emptyset, \{a, b\}, \{c\}, \Omega\}$.)

6.2 **(homework)** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space where $\Omega = [0, 1] \times [0, 1]$, $\mathcal{F}$ is the Borel $\sigma$-algebra and $\mathbb{P}$ is the Lebesgue measure on $\omega$ (restricted to $\mathcal{F}$). Let $\mathcal{G}$ be the $\sigma$-algebra

$$\mathcal{G} = \{B \times [0, 1] \mid B \subset [0, 1] \text{ is a Borel set}\}.$$

Let $X : \Omega \to \mathbb{R}$ be the random variable $X(x, y) = x(x + y)$. Calculate $\mathbb{E}(X|\mathcal{G})$.

**Solution:** $Y := \mathbb{E}(X|\mathcal{G})$ is a random variable $Y : \Omega \to \mathbb{R}$ which is $\mathcal{G}$-measurable, so $Y = Y(x, y)$ depends on $x$ only. (This is because events in $\mathcal{G}$ contain entire vertical line segments, so for any $x$, $y_1$ and $y_2$, if we set $c := Y(x, y_1)$, then $Y^{-1}(\{c\})$ has to contain $(x, y_2)$ in order to be $\mathcal{G}$-meausrable, so $Y(x, y_2) = Y(x, y_1)$.) So there is some $f : [0, 1] \to \mathbb{R}$ such that $Y(x, y) = f(x)$ for every $(x, y)$.

To find $f$, let $A_b := [0, b] \times [0, 1] \in \mathcal{G}$. The definition of the conditional expectation says that $\int_{A_b} X \,\mathrm{d}\mathbb{P} = \int_{A_b} Y \,\mathrm{d}\mathbb{P}$, so we write out the two sides:

$$\int_{A_b} X \,\mathrm{d}\mathbb{P} = \int_0^b \left[\int_0^1 x(x + y) \,\mathrm{d}y\right] \mathrm{d}x = \int_0^b x\left(x + \frac{1}{2}\right) \mathrm{d}x = \int_0^b x^2 + \frac{x}{2} \,\mathrm{d}x,$$
$$\int_{A_b} Y \,\mathrm{d}\mathbb{P} = \int_0^b f(x) \,\mathrm{d}x.$$

These have to be equal for every $b$, so

$$Y(x, y) = f(x) = x^2 + \frac{x}{2}$$

for almost every $(x, y) \in \Omega$.

6.3 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space where $\Omega = [0, 1] \times [0, 1]$, $\mathcal{F}$ is the Borel $\sigma$-algebra and $\mathbb{P}$ is the Lebesgue measure on $\omega$ (restricted to $\mathcal{F}$). Let $\mathcal{G}$ be the $\sigma$-algebra

$$\mathcal{G} = \{[0, 1] \times B \mid B \subset [0, 1] \text{ is a Borel set}\}.$$

Let $X : \Omega \to \mathbb{R}$ be the random variable $X(x, y) = x^2 + y^2$. Calculate $\mathbb{E}(X|\mathcal{G})$.

6.4 Let $\xi$ and $\eta$ be independent random variables uniformly distributed on $(0, 1)$. Let $X = \xi\eta$ and $Y = \xi/\eta$. Calcualte $\mathbb{E}(X|Y)$.

6.5 Durrett [1], Exercise 5.1.1

6.6 Durrett [1], Exercise 5.1.3

6.7 Durrett [1], Exercise 5.1.4

6.8 **(homework)** Durrett [1], Exercise 5.1.6

**Solution:** More or less anything will do, as long as $\mathcal{F}_1 \not\subset \mathcal{F}_2$ and $\mathcal{F}_2 \not\subset \mathcal{F}_1$. For example, let $\mathbb{P}$ be the uniform measure, $\mathcal{F}_1 = \{\emptyset, \{a,b\}, \{c\}, \Omega\}$, $\mathcal{F}_2 = \{\emptyset, \{a\}, \{b,c\}, \Omega\}$, and

$$X(\omega) = \begin{cases} 0 & \text{if } \omega = a \text{ or } \omega = b \\ 1 & \text{if } \omega = c \end{cases}.$$

Then $\mathbb{E}(X|\mathcal{F}_1) = X$ because $X$ is $\mathcal{F}_1$-measurable, and from the definition of conditional expectation we immediately get that

$$Y := \mathbb{E}(X|\mathcal{F}_2)(\omega) = \begin{cases} X(a) = 0 & \text{if } \omega = a \\ \frac{X(b)+X(c)}{2} = \frac{1}{2} & \text{if } \omega = b \text{ or } \omega = c \end{cases}.$$

This implies that

$$\mathbb{E}(\mathbb{E}(X|\mathcal{F}_1)|\mathcal{F}_2)(a) = \mathbb{E}(X|\mathcal{F}_2)(a) = Y(a) = 0,$$

but

$$\mathbb{E}(\mathbb{E}(X|\mathcal{F}_2)|\mathcal{F}_1)(a) = \mathbb{E}(Y|\mathcal{F}_1)(a) = \frac{Y(a)+Y(b)}{2} = \frac{1}{4}.$$

So at least for $\omega = a$,

$$\mathbb{E}(\mathbb{E}(X|\mathcal{F}_1)|\mathcal{F}_2)(\omega) \neq \mathbb{E}(\mathbb{E}(X|\mathcal{F}_2)|\mathcal{F}_1)(\omega).$$

6.9 Durrett [1], Exercise 5.2.1

6.10 Durrett [1], Exercise 5.2.3

6.11 Durrett [1], Exercise 5.2.4

6.12 **(homework)** Let $X_n$ be a martingale w.r.t. the filtration $\mathcal{F}_n$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let the random variable $\tau : \Omega \to \mathbb{N}$ be a *stopping time*, meaning

$$\{\tau = k\} := \{\omega \in \Omega \mid \tau(\omega) = k\} \in \mathcal{F}_k \quad \text{for every } k.$$

Using the notation $a \wedge b := \min\{a, b\}$, we introduce the process

$$Y_n := X_{\tau \wedge n} = \begin{cases} X_n & \text{if } n < \tau, \\ X_\tau & \text{if } n \geq \tau. \end{cases}$$

Show that $Y_n$ is also a martingale w.r.t. $\mathcal{F}_n$. (Hint: $Y_n$ is the fortune of a gambler with a certain strategy.)

**Solution 1 (painful):** We check the definition.

a.) For any $B \subset \mathbb{R}$ measurable, $\{Y_n \in B\} = (\{n < \tau\} \cap \{X_n \in B\}) \cup (\{\tau \leq n\} \cap \{X_\tau \in B\}) \in \mathcal{F}_n$, so $Y_n$ is adapted.

b.) $|Y_n| = |X_{n\wedge\tau}| \leq |X_1| + |X_2| + \cdots + |X_n|$, so $\mathbb{E}|Y_n| \leq \mathbb{E}|X_1| + \cdots + \mathbb{E}|X_n| < \infty$, so $Y_n$ is integrable.

c.) The essence is to check that $\mathbb{E}(Y_{n+1}|\mathcal{F}_n) = Y_n$. We show this by checking the definition of the conditional expectation. We have seen that $Y_n \in \mathcal{F}_n$, so we only need that

$$\int_B Y_n \, \mathrm{d}\mathbb{P} = \int_B Y_{n+1} \, \mathrm{d}\mathbb{P} \quad \text{for } B \in \mathcal{F}_n.$$

For this purpose, let $A = \{\tau \leq n\}$, so $A \in \mathcal{F}_n$.

- On the event $A$ we have $\tau \le n$, so $n \wedge \tau = \tau$, so $Y_n = X_\tau$. Also, we have $\tau \le n+1$, so $Y_{n+1} = X_\tau$ as well. All in all, on the event $A$ we have $Y_{n+1} = Y_n$.
- On the event $A^c$ we have $\tau > n$, so $n \wedge \tau = n$, so $Y_n = X_n$. But we also have $\tau \ge n+1$, so $n+1 \wedge \tau = n+1$ and $Y_{n+1} = X_{n+1}$ (on $A^c$).

Now we take $B \in \mathcal{F}_n$ and write

$$\int_B Y_n \,\mathrm{d}\mathbb{P} = \int_{B\cap A} Y_n \,\mathrm{d}\mathbb{P} + \int_{B\setminus A} Y_n \,\mathrm{d}\mathbb{P}.$$

In the first term $Y_n = Y_{n+1}$, since $B \cap A \subset A$. In the second term $Y_n = X_n$, since $B \setminus A \subset A^c$. Now we use that $B \setminus A \in \mathcal{F}_n$ and $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$ to get

$$\int_{B\setminus A} Y_n \,\mathrm{d}\mathbb{P} = \int_{B\setminus A} X_n \,\mathrm{d}\mathbb{P} = \int_{B\setminus A} X_{n+1} \,\mathrm{d}\mathbb{P}.$$

We use that $X_{n+1} = Y_{n+1}$ on $B \setminus A \subset A^c$ to conclude that $\int_{B\setminus A} Y_n \,\mathrm{d}\mathbb{P} = \int_{B\setminus A} Y_{n+1} \,\mathrm{d}\mathbb{P}$. Putting these together, we get

$$\int_B Y_n \,\mathrm{d}\mathbb{P} = \int_{B\cap A} Y_n \,\mathrm{d}\mathbb{P} + \int_{B\setminus A} Y_n \,\mathrm{d}\mathbb{P} = \int_{B\cap A} Y_{n+1} \,\mathrm{d}\mathbb{P} + \int_{B\setminus A} Y_{n+1} \,\mathrm{d}\mathbb{P} = \int_B Y_{n+1} \,\mathrm{d}\mathbb{P}.$$

**Solution 2 (elegant):** Think of $X_n$ as a stock price. An investor buys one stock at time 0 and sells it at time $\tau$. So the number of stocks she holds is

$$H_n := \begin{cases} 1 & \text{if } n \le \tau \\ 0 & \text{if } n > \tau \end{cases}.$$

Since $\tau$ is a stopping time, $\{\tau \le n\} \in \mathcal{F}_n$, so $\{H_n = 0\} = \{\tau \le n-1\} \in \mathcal{F}_{n-1}$, meaning that $H_n$ is predictable. $H_n$ is also bounded, so we know that the discrete stochastic integral

$$(H \cdot X)_n := \sum_{m=1}^{n} H_m (X_m - X_{m-1})$$

is also a martingale. But

$$Y_n = X_0 + (H \cdot X)_n$$

and $X_0 \in \mathcal{F}_0 \subset \mathcal{F}_n$ for every $n$, so $Y_n$ is also a martingale.

6.13 **(homework)** Let $a, b \in \mathbb{Z}$ with $a < 0 < b$. Let $S_n$ be a simple symmetric random walk with $S_0 = 0$ and let $\tau$ be the first hitting time for $\{a, b\}$. Apply the optional stopping theorem to the martingale $S_n$ to find the hitting probabilities $p_a = \mathbb{P}(S_\tau = a)$ and $p_b = \mathbb{P}(S_\tau = b)$.

**Solution:** We will heavily use the fact that the random walk hits the set $\{a, b\}$ almost surely, meaning that $\mathbb{P}(\tau < \infty) = 1$ and $p_a + p_b = 1$. Let's believe this for a moment.

The martingale $S_n$ has bounded increments and the stopping time $\tau$ is almost surely finite, so the optional stopping theorem says that $\mathbb{E}S_\tau = \mathbb{E}S_0$. (The second sufficient condition in the theorem is satisfied.) Now $\mathbb{E}S_\tau = ap_a + bp_b$ and $\mathbb{E}S_0 = 0$, so $p_a$ and $p_b$ satisfy the following system of linear equations:

$$\begin{cases} ap_a + bp_b &= 0 \\ p_a + p_b &= 1 \end{cases}$$

The unique solution is

$$p_a = \frac{b}{b-a}$$
$$p_b = \frac{-a}{b-a}.$$

We are left to show that $\mathbb{P}(\tau < \infty) = 1$. There are many ways to do this. For example: The stopped martingale $S_{\tau \wedge n}$ is also a martingale by Exercise 12. It is also bounded by construction, so the martingale convergence theorem says that it is almost surely convergent. Since it is integer valued, it can only be convergent by being eventually constant, maning that the walk has to reach $a$ or $b$.

6.14 Let $p \in (0, 1)$ be fixed, and let $q = 1 - p$. A frog performs a (discrete time) random walk on the 1-dimensional lattice $\mathbb{Z}$ the following way:

The initial position is $X_0 = 0$. The frog jumps 1 step up with probability $p$ and jumps 1 step down with probability $q$ at each time step, independently of what happened before, until it reaches either the point $a = -10$ or the point $b = +30$, which are *sticky*: if the frog reaches one of them, it stays there forever.

Let $X_n$ denote the position of the frog after $n$ steps (for $n = 0, 1, 2, \dots$).

a.) Show that $Y_n := \left(\frac{q}{p}\right)^{X_n}$ is a martingale (w.r.t. the natural filtration).

b.) Show that $Y_n$ converges almost surely to some limiting random variable $Y_\infty$. What are the possible values of $Y_\infty$?

c.) How much is $\mathbb{E}Y_\infty$ and why?

d.) Suppose now that $p \neq \frac{1}{2}$. Use the previous results to calculate the probability that the frog eventually gets stuck at the point $a = -10$.

6.15 Let $0 \le p \le 1$ and $q = 1 - p$. Let $X_1, X_2, \dots$ be i.i.d. with $\mathbb{P}(X_i = -1) = q$ and $\mathbb{P}(X_i = 1) = p$. For $n = 0, 1, \dots$ let $S_n = X_1 + \dots + X_n$. So $S_n$ is a simple asymmetric random walk starting from $S_0 = 0$. (Symmetric if $p = \frac{1}{2}$.) Show that $M_n := S_n - n(p - q)$ is a martingale (w.r.t. the natural filtration).

For $p \neq q$, use this to find the expectation of the time when the frog of Exercise 14 gets stuck.

6.16 Let $X_1, X_2, \dots$ be i.i.d. with $\mathbb{P}(X_i = -1) = \mathbb{P}(X_i = 1) = \frac{1}{2}$. For $n = 0, 1, \dots$ let $S_n = X_1 + \dots + X_n$. So $S_n$ is a simple symmetric random walk starting from $S_0 = 0$.

a.) Show that $S_n^2 - n$ is a martingale (w.r.t. the natural filtration). *This is a special case of Durrett [1], Exercise 5.2.6. You can also solve that – it' not any harder.*

b.) Use this and the result of Exercise 13 to find the expectation of the stopping time when the walk first reaches either $-10$ or 30.

c.) How about the expectation of the stopping time when the walk first reaches 30?

6.17 Let $\mathcal{F}_n$ be a filtration and $X$ any random varibale with $\mathbb{E}|X| < \infty$. Let $X_n = \mathbb{E}(X|\mathcal{F}_n)$.

a.) Show that $X_n$ is a martingale w.r.t. $\mathcal{F}_n$.

b.) Show that $X_n$ converges almost surely to some limit $X_\infty$.

c.) Give a specific example when $X_\infty \neq X$.

d.) Give a specific example when $X_\infty = X$.

# References

[1] Durrett, R. *Probability: Theory and Examples.* **4th** edition, Cambridge University Press (2010)