

Exchanging integral and limit

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Let $(X, \mathcal{F}, \mu)$ be a measure space, and let

$f_1, f_2, \dots; f_0: X \rightarrow \mathbb{R}$ (or possibly $X \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$) be (measurable) functions.

Q: What do we mean if we say $f_n \xrightarrow{n \rightarrow \infty} f_0$?

A: This can mean many different things. It should definitely mean that f_n is "close" to f_0 if n is big, but "close" ^{the function} can have different meanings.

Some examples:

Def: ① $f_n \xrightarrow{n \rightarrow \infty} f_0$ pointwise, if $\forall x \in X \ f_n(x) \rightarrow f_0(x)$.

[Note that $f_n(x)$ is just a sequence of numbers and $f_0(x)$ is also just a number, so convergence of these requires no explanation.]

② $f_n \xrightarrow{n \rightarrow \infty} f_0$ almost everywhere, if $f_n(x) \xrightarrow{n \rightarrow \infty} f_0(x)$ for μ -almost every $x \in X$ — that is, $\mu(\{x \in X \mid f_n(x) \not\rightarrow f_0(x)\}) = 0$.

These notions ① and ② are essentially the same, since in measure theory we "never" care about 0-measure sets. (Sooner or later we will integrate, and

0-measure sets make no difference in the integral.

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So in practice we always talk about almost sure convergence, even if we have pointwise convergence in mind.

③ ~~And~~ If μ is a probability (that is: $\mu(X)=1$), then "almost everywhere" convergence is also called almost sure convergence:

$f_n \xrightarrow{n \rightarrow \infty} f_\infty$ almost surely if $\mu(\{x \in X \mid f_n(x) \rightarrow f_\infty(x)\}) = 1$.

④ $f_n \xrightarrow{n \rightarrow \infty} f_\infty$ in the L^1 sense if

$$\underbrace{\int_X |f_n - f_\infty| d\mu}_{\text{a sequence of numbers}} \xrightarrow{n \rightarrow \infty} 0$$

⑤ $f_n \xrightarrow{n \rightarrow \infty} f_\infty$ in the L^2 sense if

$$\underbrace{\int_X |f_n - f_\infty|^2 d\mu}_{\text{a sequence of numbers}} \xrightarrow{n \rightarrow \infty} 0$$

Remark: Later / in Functional Analysis we will have

" L^1 norm", " L^2 norm", " L^1 space" and " L^2 space", but for now, these are just phrases for the type of convergence.

Question: Are these notions of convergence really all different? Do at least some of them imply others?

Most importantly: Does ~~not~~ pointwise (or almost everywhere) convergence imply L^1 convergence?

Or, to put it simpler: If $f_n \rightarrow f$ almost everywhere, does this imply that $\underbrace{\int_X f_n d\mu}_{\text{a sequence of numbers}} \xrightarrow{n \rightarrow \infty} \underbrace{\int_X f d\mu}_{\text{a number}}$

Answers:

YES, they are all truly different.

NO, none of them implies any other one

{ How:
find
examples

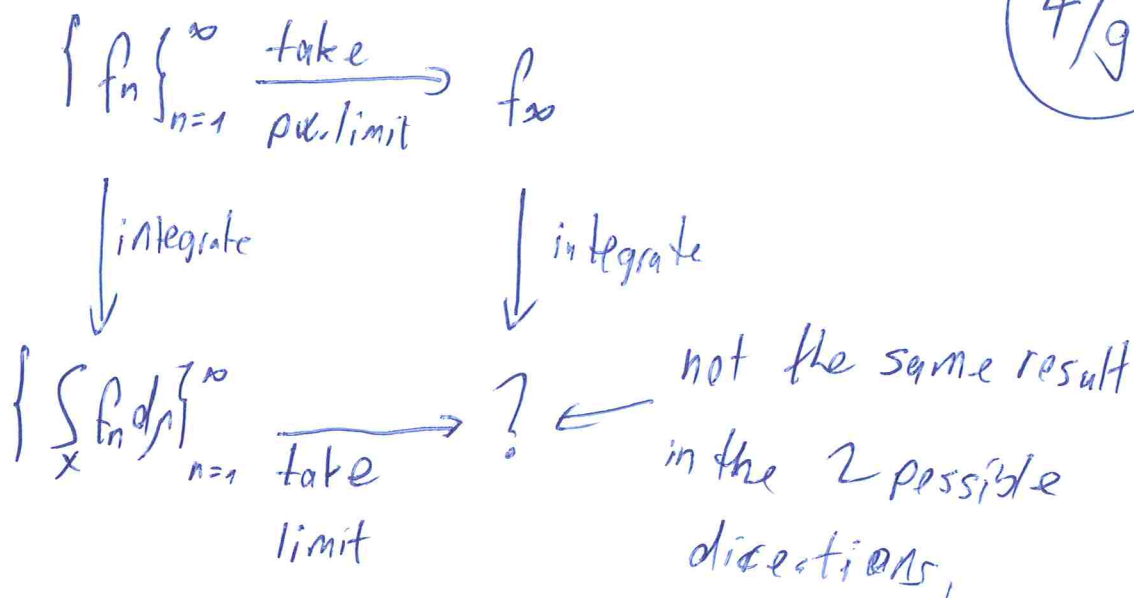
Most importantly: NO, NO, NO, pointwise convergence does NOT imply convergence of the integrals.

In other words:
in general

$$\underbrace{\int_X \lim_{n \rightarrow \infty} f_n(x) d\mu(x)}_{\text{integral and}} \neq \underbrace{\lim_{n \rightarrow \infty} \int_X f_n(x) d\mu(x)}_{\text{limit can NOT be exchanged}}$$

Or:

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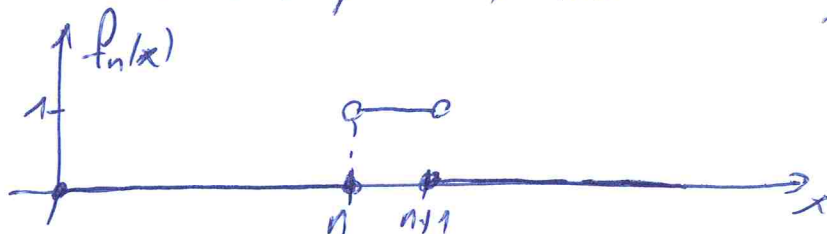
"the diagram does not commute".

Important: This is shocking bad news, because we very often want to exchange the integral & the limit, and we very often expect $\int_X \lim_n f_n d\mu = \lim_n \int_X f_n d\mu$.

So Examples $\approx$ What can go wrong?

For simplicity, we can always take $f_0 \equiv 0$.

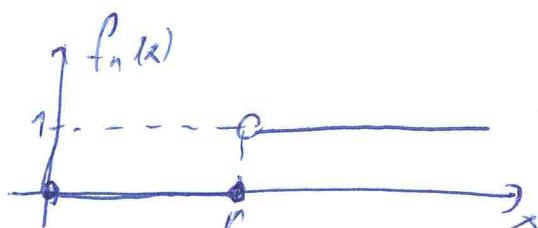
① Take $X = \mathbb{R}^+$, $\mu = \text{Leb}$, and



Then $\int_X f_n d\mu = 1$ for $\forall n$,
but although $f_n(x) \xrightarrow{n \rightarrow \infty} 0$ for $\forall x$

②

$X = \mathbb{R}^+$, $\mu = \text{Leb}$,

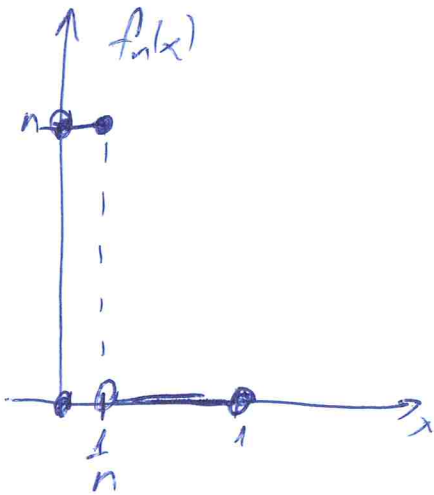


So $f_n(x) \rightarrow 0$ $\forall x$
but $\int_X f_n d\mu = \infty$ $\forall n$

(convergence is now monotone, but this doesn't help)

③

$$X = [0, 1], \mu = \text{Leb},$$



Again $f_n(x) \xrightarrow{n \rightarrow \infty} 0$ for $\forall x$,

$$\text{but } \int_X f_n d\mu = 1 \text{ for } \forall n.$$

Now $\mu(x) = 1 < \infty$, but this doesn't help.

Project: We do NOT accept the answer "No, sorry".

We want to exchange the integral with the limit, so we want positive statements, at the cost of extra assumptions, of course.

Alternatively: we are looking for sufficient conditions under which $\int \lim = \lim \int$.

Remark 1: The best would be a sufficient & necessary condition, but this we will not have.

However we give 3 different sufficient conditions. These are a "morally necessary" in the sense that if none of them works, then it's time to suspect that $\int \lim f_n d\mu$ and $\lim \int f_n d\mu$ are really different. (or

there is some very fortunate constellation of the stars.) 6/9

Remark 2: Of course, the sufficient conditions we will have are designed to rule out the counterexamples above.

Theorem (Monotone convergence theorem)

Let $(X, \mathcal{F}, \mu)$ be a measure space, let $f_1, f_2, f_3, \dots : X \rightarrow \mathbb{R}$ ^{measurable} ~~meas.~~

① Assume that $f_1, f_2, \dots$ are all non-negative: $f_n(x) \geq 0 \ \forall x, n$

- The sequence $f_1, f_2, \dots$ is pointwise increasing, meaning $f_{n+1}(x) \geq f_n(x)$ for $\forall x, n$.

[Then the pointwise limit $\lim_{n \rightarrow \infty} f_n(x)$ exists automatically, and is measurable automatically.]

$$\text{Then } \int_X \lim_{n \rightarrow \infty} f_n(x) d\mu(x) = \lim_{n \rightarrow \infty} \int_X f_n(x) d\mu(x).$$

② Alternatively, assume that

- $f_1, f_2, \dots \geq 0$
- The sequence $f_1, f_2, \dots$ is pointwise decreasing
- $\int_X f_1 d\mu < \infty$

Then again, the limit and the integral can be exchanged.

Remark: Of course, $\forall x$ could be relaxed to μ -a.e. x in all the assumptions. (7/9)

Theorem (Lebesgue dominated convergence theorem)

Let $(X, \mathcal{F}, \mu)$ be a measure space.

Let $f_1, f_2, \dots; f_\infty$ and $g: X \rightarrow \mathbb{R}$ measurable functions.

Assume that

- $f_n \xrightarrow{n \rightarrow \infty} f_\infty$ almost everywhere
 - $|f_n(x)| \leq g(x)$ for every n and (almost) every x
 - $\int_X g d\mu < \infty$
- } so

g is a common integrable dominating function of the f_n

Then $\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f_\infty d\mu$ (both sides exist and are finite)

Remark: This is a strange theorem, since the function g of the assumptions, does not show up in the statements; the only role of g is that it has to exist.

Now if some common dominating function g is integrable, then surely the smallest of all common dominating functions, $g_{\min}(x) := \sup\{|f_n(x)| : n \in \mathbb{N}\}$ must also be integrable.

So the theorem could be rephrased as:

- Assume $f_n \rightarrow f_\infty$ almost everywhere;
- Assume $\int_X \left(\sup_n |f_n(x)| \right) d\mu(x) < \infty$;

$$\text{Then } \int_X \lim_n f_n d\mu = \lim_n \int_X f_n d\mu.$$

However, in practice, we seldom can / need to find the smallest dominating function: any integrable g , found with any intuition, will do. So the form of the theorem as 1st stated is closer to the applications in spirit.

Theorem (Fatou's lemma)

Remark: This time, instead of putting stronger assumptions, we weaken the statement, which allows for very weak assumptions.

Let $(X, \mathcal{F}, \mu)$ be a measure space and

Let $f_1, f_2, f_3, \dots: X \rightarrow \mathbb{R}$ be $\begin{cases} \bullet \text{ measurable} \\ \bullet \text{ and non-negative} \end{cases}$ (and that's all)

Then

$$\int_X \liminf_{n \rightarrow \infty} f_n(x) d\mu(x) \leq \liminf_{n \rightarrow \infty} \int_X f_n(x) d\mu(x)$$

the limit is not guaranteed to exist

equality is not guaranteed.

Note: This lemma is miraculously often useful when the other 2 theorems don't apply.

Applications: see later.