

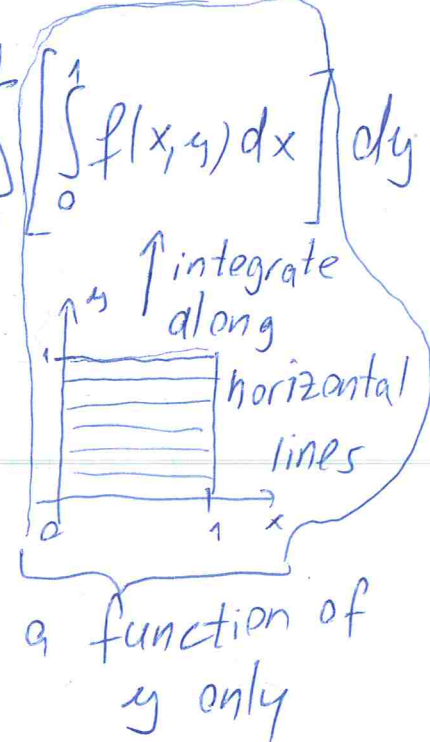
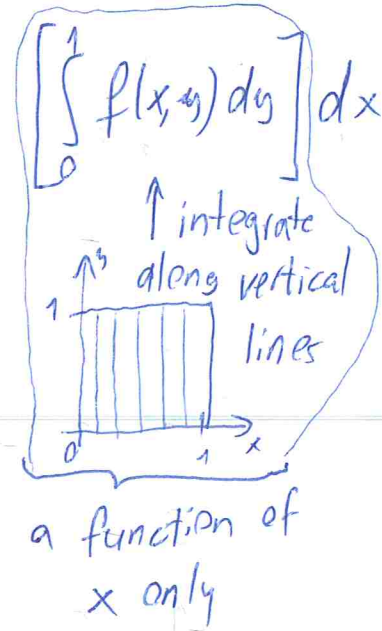
Exchanging integrals — product measure, Fubini's theorem

Consider a function of 2 variables: $f(x, y) \in \mathbb{R}$
and integrate w.r.t. both variables

1/7

Say, $f: [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ or $g: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ or $q: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$.

Question: Is it always true that

$$\int_0^1 \left[\int_0^1 f(x, y) dy \right] dx = \int_0^1 \left[\int_0^1 f(x, y) dx \right] dy \quad ?$$


Similarly:

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} g(x, y) dy \right] dx \stackrel{?}{=} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} g(x, y) dx \right] dy$$

$$\sum_{k=0}^{\infty} \left[\sum_{l=0}^{\infty} a_{k,l} \right] \stackrel{?}{=} \sum_{l=0}^{\infty} \left[\sum_{k=0}^{\infty} a_{k,l} \right]$$

[Remember: Sum = integral w.r.t. counting measure
Riemann integral $\approx$ integral w.r.t. Lebesgue measure]

Answer: In general no

2/4

1. Easy example: $a_{k,e} := k - e$

then $\forall$ fixed e , $\sum_{k=0}^{\infty} a_{k,e} = -e + (-e+1) + (-e+2) + \dots = -\infty$

$$\text{So } \sum_{e=0}^{\infty} \left[\sum_{k=0}^{\infty} a_{k,e} \right] = \infty + \infty + \infty + \dots = \infty$$

but for $\forall$ fixed k , $\sum_{e=0}^{\infty} a_{k,e} = k + (k-1) + (k-2) + \dots = -\infty$

NOT EQUAL

$$\text{So } \sum_{k=0}^{\infty} \left[\sum_{e=0}^{\infty} a_{k,e} \right] = (-\infty) + (-\infty) + \dots = -\infty$$

2. Still easy example, if you don't like ∞ 's:

$$a_{k,e} := \begin{cases} 1, & \text{if } k = e+1 \\ -1, & \text{if } k = e-1 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{array}{cccccc} & e & & & & \\ & \uparrow & & & & \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \\ \hline & 0 & 0 & 0 & -1 & 0 \\ & 0 & 0 & -1 & 0 & 1 \\ & 0 & -1 & 0 & 1 & 0 \\ & -1 & 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 & 0 & 0 \end{array}$$

$$\text{Then } \sum_{e=0}^{\infty} a_{k,e} = \begin{cases} -1 & \text{if } k=0 \\ 0 & \text{if not} \end{cases}$$

$$\text{So } \sum_{k=0}^{\infty} \left[\sum_{e=0}^{\infty} a_{k,e} \right] = -1 + 0 + 0 + \dots = -1$$

NOT EQUAL

$$\text{and similarly } \sum_{e=0}^{\infty} \left[\sum_{k=0}^{\infty} a_{k,e} \right] = 1 + 0 + 0 + \dots = +1$$

3. Easy to convert these to examples $g: [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$

4. HW: find examples $f: [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ (still not hard)

Project: We are unhappy with this answer; 3/4
 We do want to exchange integrals all the time,
 So we want $\rightarrow$ positive statements (under
 extra assumptions, of course)
 $\searrow$ to understand what can go
 wrong

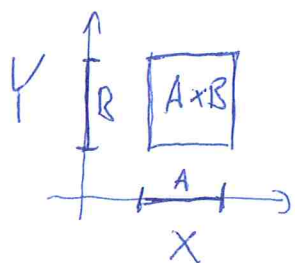
Remark: Examples 1 and 2 already suggest the solution:
 When we sum up infinitely many numbers, the order
 of summation matters if the sum is conditionally
convergent, but does not matter if it's absolutely
convergent.

Preparation: functions of 2 variables in measure theory:
Product spaces.

Def: let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be measurable spaces.

On the Cartesian product $X \times Y$, consider the

σ -algebra $\mathcal{F} \otimes \mathcal{G} := \sigma(\{A \times B \mid A \in \mathcal{F}, B \in \mathcal{G}\})$



We want rectangles to be measurable
 the generated σ -algebra is the
 smallest one that ensures this

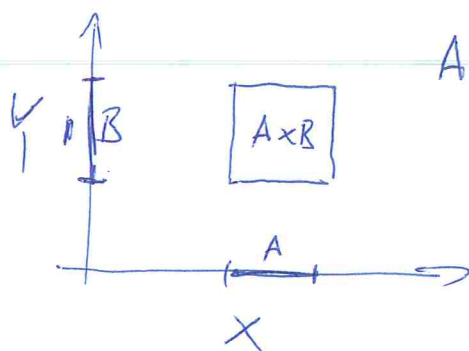
Then $\mathcal{F} \otimes \mathcal{G}$ is called the product σ -algebra,

and $(X \times Y, \mathcal{F} \otimes \mathcal{G})$ is called the product measurable space.

~~To define~~ Now if μ is a measure on $(X, \mathcal{F})$
and ν a measure on $(Y, \mathcal{G})$,

then it is natural to assign the weight
 $g(A \times B) := \mu(A) \cdot \nu(B)$ to the rectangle.

Example:



$$\text{Area}(A \times B) = \text{length}(A) \cdot \text{length}(B)$$

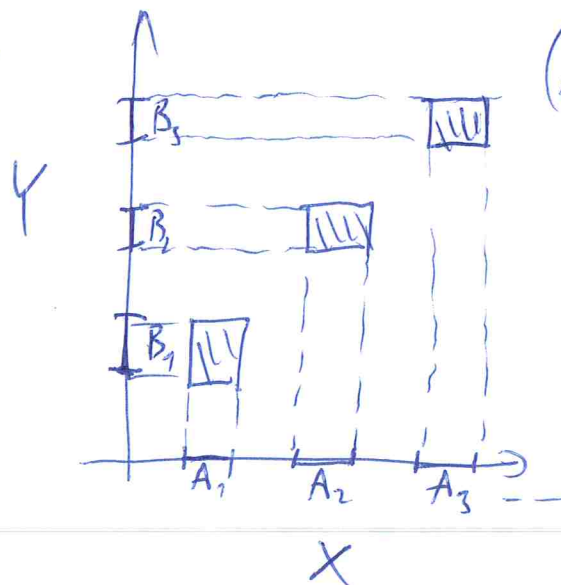
Def: let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ be measure spaces,
and let g be a measure on $(X \times Y, \mathcal{F} \otimes \mathcal{G})$.
This g is called a product measure, and the
space $(X \times Y, \mathcal{F} \otimes \mathcal{G}, g)$ is called a product measure
space, if $\forall A \in \mathcal{F}, B \in \mathcal{G} \quad g(A \times B) = \mu(A) \cdot \nu(B)$.

Notation: $g = \mu \otimes \nu$

Question: Does this exist? Is this unique?

Warning: We only gave the value of g on rectangles, and $\mathcal{F} \otimes \mathcal{G}$ contains many more measurable sets,

e.g.



$(A_1 \times B_1) \cup (A_2 \times B_2) \cup (A_3 \times B_3) \cup \dots$
is also $\in \mathcal{F} \otimes \mathcal{G}$, although it is not a rectangle.

Is it possible to extend g from rectangles to all of $\mathcal{F} \otimes \mathcal{G}$? If yes, is it possible in only 1 way?

Good news: Yes, a product measure always exists.

[The standard machinery of measure theory "how to extend your wishlist into a measure" (not discussed in this course) works. (Keyword: Caratheodory construction)]

Bad news: No, it's not always unique (i)

Good news 2: For reasonable μ and ν it is unique:

(6/7)

Theorem: If μ and ν are σ -finite measures,
then $\mu \otimes \nu$ exists and is unique.

[Remember: a measure which is not σ -finite is
insanely big: avoid it if possible.]

Now let $f: X \times Y \rightarrow \mathbb{R}$, and we want to integrate it
— w.r.t. $\mu \times \nu$, of course.

Q1: Can we integrate out the variables 1 by 1?

Is it true that

$$\begin{aligned} \int_{X \times Y} f d(\mu \otimes \nu) &= \int_X \left[\underbrace{\int_Y f(x, y) d\nu(y)}_{\text{a function of } x \text{ only}} \right] d\mu(x) \\ &= \int_Y \left[\underbrace{\int_X f(x, y) d\mu(x)}_{\text{a function of } y \text{ only}} \right] d\nu(y) \end{aligned} \quad ?$$

Answer: No, we have seen counterexamples.

Q2: Any positive statements?

Answer 2: YES, it works, if $f \geq 0$ $\leftarrow$ Tonelli theorem
 or f is integrable $\leftarrow$ Fubini theorem
 AND μ, ν are reasonable

Theorem (Fubini-Tonelli)

Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ be σ -finite measure spaces,

and let $f: X \times Y \rightarrow \mathbb{R} \cup [-\infty, \infty]$ be measurable.

Assume also that $f \geq 0$

or that $\int_{X \times Y} |f| d(\mu \otimes \nu) < \infty$.

$\uparrow$
 w.r.t. $\mathcal{F} \otimes \mathcal{G}$,
 of course.

Then

$$\int_{X \times Y} f d(\mu \otimes \nu) = \int_X \int_Y f(x, y) d\nu(y) d\mu(x) = \int_Y \int_X f(x, y) d\mu(x) d\nu(y)$$

Remark: The theorem also says that all 3 expressions in the equality make sense. For example, $\int_Y f(x, y) d\nu(y)$ may not exist for every x , but it is guaranteed to exist for μ -almost every $x \in X$, and $x \mapsto \int_Y f(x, y) d\nu(y)$ is guaranteed to be $\mathcal{F}$ -measurable.