Tools of Modern Probability
exam exercise sheet, 08.01.2020
(working time: 90 minutes)

1. Describe the asymptotic behaviour of the following sequences as $n \to \infty$:

   a.) $A_n := \int_{-1}^{1} \cos^n x \, \mathrm{d}x$

   b.) $B_n := \int_{-\infty}^{\infty} \cos^n x \, \mathrm{d}x$

   (By "describing the asymptotic behaviour" I mean: find sequences a_n and b_n given by nice simple formulas, such that $A_n \sim a_n$ and $B_n \sim b_n$.)

2. Let $X = [-1,1] \times [-1,1]$ and let μ be Lebesgue measure on X. Let $f : X \to \mathbb{R}$ be given by $f(x,y) = x + y$ and let $\nu = f_* \mu$. Calculate $\int_{\mathbb{R}} u \, \mathrm{d}\nu(u)$.

3. Let μ be a probability distribution on $\mathbb{R}$. Show that the function $f : \mathbb{R} \to \mathbb{C}$ defined by $f(t) := \int_{\mathbb{R}} e^{itx} \, \mathrm{d}\mu(x)$ is continuous.

4. Consider the Hilbert space $V = L^2(X, \mu)$ where $(X, \mathcal{F}, \mu)$ is some measure space. Let $A \in \mathcal{F}$ and consider the operator $L : V \to \mathbb{R}$ given by $Lf := \int_A f \, \mathrm{d}\mu$. Show that this operator is linear and bounded, or give a counterexample.

5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $X : \Omega \to \mathbb{R}$ integrable and $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra. Show that the conditional expectation $\mathbb{E}(X \mid \mathcal{G})$ exists. *(Show means: sketch the proof.)*

6. Let X and Y be independent, standard Gaussian random variables. Find $\mathbb{E}(X \mid X + 2Y)$.

7. Let X and Y be independent random variables with *Bernoulli* $\left(\frac{2}{3}\right)$ distribution. Find $\mathbb{E}(X \mid X + 2Y)$.

8. Let X_n have geometric distribution with parameter $p_n = \frac{1}{n}$ and let $Y_n = \frac{X_n}{n}$. Find the weak limit of Y_n. *(Hint: find the pointwise limit of the distribution function sequence.)*