Tools of Modern Probability
exam exercise sheet, 21.12.2022
(working time: 90 minutes)

Every exercise is worth 12 points. Every student should *choose* 5 exercises to solve. If more than 5 are solved, I only take the best 5 into account. The maximum total score is 60. Solve easy exercises if you want a good grade. Solve hard ones if you want me to be proud of you.

1. Describe the asymptotic behaviour of the following sequences as $n \to \infty$:

   a.) $A_n := \int_0^1 (x - x^3)^n \, \mathrm{d}x$

   b.) $B_n := \int_{-1}^1 (x - x^3)^n \, \mathrm{d}x$

   (By "describing the asymptotic behaviour" I mean: find sequences a_n and b_n given by nice simple formulas, such that $A_n \sim a_n$ and $B_n \sim b_n$.)

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be

$$f(x_1, x_2, \dots, x_n) = \begin{cases} x_1^2 + x_2^2 + \dots + x_n^2 & \text{if } x_1^2 + x_2^2 + \dots + x_n^2 \leq 1 \\ 0 & \text{if not} \end{cases} .$$

   Calculate $\int_{\mathbb{R}^n} f(x_1, \dots, x_n) \, \mathrm{d}x_1 \dots \mathrm{d}x_n$.

   Help: the surface volume of the d-dimensional unit sphere is $c_d = 2\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$.

3. Let χ be counting measure on $\mathbb{Z}$ and let $f(x) = x^2$. Let ν have density f w.r.t. χ and let $\mu = f_* \nu$. Calculate $\int f \, \mathrm{d}\mu$.

4. Prove that if $X_n \Rightarrow X$ then $\mathbb{E}X_n \to \mathbb{E}X$, or give a counterexample. Here $\Rightarrow$ denotes weak convergence of random variables.

5. Prove that if $f_1, f_2, f_3, \dots$ and $f : \mathbb{R} \to \mathbb{R}^+$ are Lebesgue integrable functions and $f_n(x) \to f(x)$ for every $x \in \mathbb{R}$, then $\int_{-\infty}^y f_n(x) \, \mathrm{d}x \to \int_{-\infty}^y f(x) \, \mathrm{d}x$ for every $y \in \mathbb{R}$, or give a counterexample.

6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space where $\Omega = [0, 1] \times [0, 1]$ and $\mathcal{F}$ is the Borel σ-algebra. Let $X, Y : \Omega \to \mathbb{R}$ be the natural projections: $X(x, y) = x$ and $Y(x, y) = y$. Let $\mathcal{G} = \sigma(X)$ be the generated σ-algebra. Calculate $\mathbb{E}(Y \mid \mathcal{G})$ if

   a.) $\mathbb{P}$ is Lebesgue measure on Ω,

   b.) $\mathbb{P}$ has density $\rho(x, y) = x + y$ w.r.t. Lebesgue measure on Ω.

7. Let X, Y be independent and uniformly distributed on $[-1, 2]$. Let $Z = X + Y$. Calculate $\mathbb{E}(Z^2 | Z^3)$.

8. Let X, Y be jointly Gaussian random variables with covariance matrix

$$C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} .$$

   Calculate $\mathbb{E}(X \mid X + Y)$.