Tools of Modern Probability
exam exercise sheet, 22.01.2020
(working time: 90 minutes)

Every exercise is worth 14 points. Every student should *choose* 5 exercises to solve. If more than 5 are solved, I only take the best 5 into account. The maximum total score is 70. Solve easy exercises if you want a good grade. Solve hard ones if you want me to be proud of you.

1. Describe the asymptotic behaviour of the following sequences as $n \to \infty$:

   a.) $A_n := \displaystyle\int_{-1}^{1} \frac{e^{nx^2}}{e^{nx^4}} \, \mathrm{d}x$

   b.) $B_n := \displaystyle\int_{0}^{\infty} x^{\frac{n+1}{2}} e^{-x} \, \mathrm{d}x$

   (By "describing the asymptotic behaviour" I mean: find sequences a_n and b_n given by nice simple formulas, such that $A_n \sim a_n$ and $B_n \sim b_n$.)

2. Let the probability measure μ have density $f(k) = \frac{c}{1+k^2}$ with respect to counting measure on $\mathbb{Z}$ (where the constant c is chosen appropriately). Let the random variable X have distribution μ. Calculate $\mathbb{E}\left(X^3\right)$.

3. We roll a fair die 10 times. Let μ be the distribution of the product of the numbers rolled. Calculate $\int_{\mathbb{R}} x^2 \, \mathrm{d}\mu(x)$.

4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X_1, X_2, \ldots$ be random variables. Assume that $X_n \to 0$ in L^1. Show that $X_n \to 0$ almost surely as well, or give a counterexample.

5. Let μ, ν be finite measures on the measurable space $(X, \mathcal{F})$ and let $\rho = \mu + \nu$. Show that there is a function $f \in L^2(X, \rho)$ such that $\int_X g \, \mathrm{d}\nu = \int_X fg \, \mathrm{d}\rho$ for all $g \in L^2(X, \rho)$, or give a counterexample.

6. Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space and let $X, Y : \Omega \to \mathbb{R}$ be $\mathcal{G}$-measurable. Assume that $\int_A X \, \mathrm{d}\mathbb{P} = \int_A Y \, \mathrm{d}\mathbb{P}$ for all $A \in \mathcal{G}$. Show that $X = Y$ almost surely.

7. Let $(X, Y) \in \mathbb{R}^2$ be a random point uniformly distributed on the unit disk $D := \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1\}$. Calculate the conditional expectation $\mathbb{E}\left(X^2 + Y^2 \,\middle|\, \frac{Y}{X}\right)$.

8. Let $\mu_1, \mu_2, \ldots$ and μ be probability distributions on $\mathbb{R}$ such that $\mu_n \Rightarrow \mu$. (That is, μ_n converges to μ weakly.) Show that there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and there are random variables $X_1, X_2, \ldots$ and $X : \Omega \to \mathbb{R}$ such that X_n has distribution μ_n, X has distribution μ and $X_n \to X$ almost surely. (*Show* means: sketch the proof.)