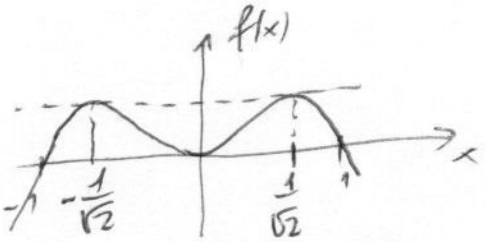
1) a) $A_n = \int_{-1}^{1} e^{n(x^2-x^4)}\,dx = \int_a^b e^{nf(x)}\,dx$ with $f(x) = x^2 - x^4$, $a=-1$, $b=1$.

So if f has a unique global maximum at x_0, then $A_n \sim e^{nA}\sqrt{\frac{2\pi}{nB}}$ with $A = f(x_0)$ and $B = -f''(x_0)$, by Laplace's theorem. However, the theorem does <u>not</u> apply directly: the maximum is taken at



$x_1 = -\frac{1}{\sqrt{2}}$ and $x_2 = +\frac{1}{\sqrt{2}}$ as well.

The values:

$f(x) = x^2 - x^4$ $\qquad f(x_1) = f(x_2) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} =: A$

$f'(x) = 2x - 4x^3$ $\qquad f'(x_1) = f'(x_2) = 0$

$f''(x) = 2 - 12x^2$ $\qquad f''(x_1) = f''(x_2) = 2 - 6 = -4 =: -B$

So let's apply Laplace's theorem twice:

$$\left.\begin{aligned} \int_{-1}^{0} e^{nf(x)}\,dx &\sim e^{nA}\sqrt{\frac{2\pi}{nB}} = e^{n/4}\sqrt{\frac{\pi}{2n}} \\ \int_{0}^{1} e^{nf(x)}\,dx &\sim e^{nA}\sqrt{\frac{2\pi}{nB}} = e^{n/4}\sqrt{\frac{\pi}{2n}} \end{aligned}\right\} \Longrightarrow \underline{\underline{A_n \sim 2e^{n/4}\sqrt{\frac{\pi}{2n}} = e^{n/4}\sqrt{\frac{2\pi}{n}}}}$$

b) $B_n = \int_0^\infty x^{\frac{n+3}{2}-1} e^{-x}\,dx = \Gamma\left(\frac{n+3}{2}\right) =$

$$= \Gamma\left(\frac{n+1}{2}+1\right) \underset{\text{approximation}}{\overset{\text{Stirling's}}{\sim}} \underline{\underline{\left(\frac{n+1}{2}\right)^{\frac{n+1}{2}} e^{-\frac{n+1}{2}} \sqrt{\frac{n+1}{2}}\sqrt{2\pi}}}$$

(2) For μ to be a probability, we need

$$1 = \mu(\mathbb{Z}) = \int_{\mathbb{Z}} f \, d\gamma = \sum_{k=-\infty}^{\infty} f(k) = c \sum_{k=-\infty}^{\infty} \frac{1}{1+k^2},$$

which is OK, since $\sum_{k=-\infty}^{\infty} \frac{1}{1+k^2} < \infty$.

~~Now~~

Now

$$\mathbb{E} X^3 = \int_{\mathbb{Z}} k^3 \, d\mu(k) = \int_{\mathbb{Z}} k^3 f(k) \, d\gamma(k) = \sum_{k=-\infty}^{\infty} k^3 \frac{c}{1+k^2} = c \sum_{k=-\infty}^{\infty} \frac{k^3}{1+k^2}.$$

This <u>does not exist</u>, since $\frac{k^3}{1+k^2} \to -\infty$ as $k \to -\infty$

$\frac{k^3}{1+k^2} \to +\infty$ as $k \to +\infty$,

so ~~but~~ the integral of both the negative part and the positive part is ∞.

(3) Let $X_1, \ldots, X_{10}$ be the numbers rolled.

These are i.i.d $\sim Uni(\{1,2,3,4,5,6\})$.

Let $P = X_1 \cdot \ldots \cdot X_{10}$, so μ is the distribution of P.

Now $\int_{\mathbb{R}} x^2 d\mu(x) = \mathbb{E}(P^2) = \mathbb{E}(X_1^2 \cdot \ldots \cdot X_{10}^2) \overset{\text{independence}}{=}$

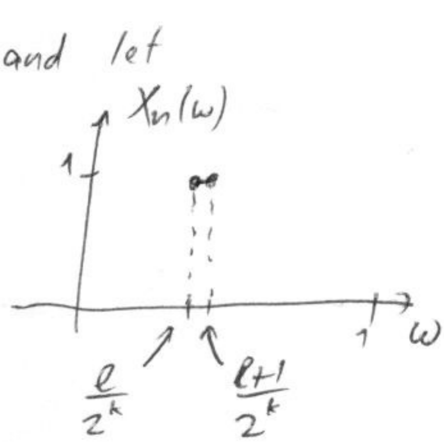
$= \mathbb{E}(X_1^2) \cdot \ldots \cdot \mathbb{E}(X_{10}^2) \underset{\text{distribution}}{\overset{\text{identical}}{=}} [\mathbb{E}(X_1^2)]^{10} = \left(\frac{1^2+2^2+3^2+4^2+5^2+6^2}{6}\right)^{10} =$

$= \underline{\underline{\left(\frac{91}{6}\right)^{10}}} \approx 6.44 \cdot 10^{11}$

(4) Let $\Omega = [0,1]$, $\mathbb{P} = \text{Leb}$. Now the function sequence in HW 2.15 (b) is a counterexample:

Write n as $n = 2^k + \ell$ with $0 \le \ell < 2^k$ and let

$$X_n(\omega) = \begin{cases} 1 & \text{if } \frac{\ell}{2^k} \le \omega < \frac{\ell+1}{2^k} \\ 0 & \text{if not} \end{cases}$$



Then $X_n(\omega)$ is divergent for every $\omega \in [0,1] = \Omega$, but

$$\|X_n - 0\|_1 = \|X_n\|_1 = \int_\Omega |X_n| \, d\mathbb{P} = \int_0^1 X_n(\omega)\, d\omega = \frac{1}{2^k} \to 0, \text{ so } X_n \xrightarrow{L^1} 0.$$

(5) This is a step from the proof (shown in class) of the Radon-Nikodym theorem:

Let $V = L^2(X, \rho)$, a Hilbert space, $\langle f, g \rangle = \int f g \, d\rho$

Let $L : V \to \mathbb{R}$ be given as $Lg := \cancel{R} \int_X g \, d\nu$.

This L is clearly linear.

It is also bounded:

$$|Lg| \le \int_X |g| \, d\nu \le \int_X |g| \, d\rho = \langle |g|, 1 \rangle \underset{\text{inequality}}{\overset{\text{Schwartz's}}{\le}} \|g\|_V \|1\|_V ,$$

and $\|1\|_V^2 = \langle 1, 1 \rangle = \int_X 1^2 \, d\rho = \rho(X) = \mu(X) + \nu(X) < \infty$ by assumption,

So $|Lg| \le K \|g\|$ with $K = \sqrt{\mu(X) + \nu(X)} < \infty$.

Now the Riest representation theorem ensures the existence of $f \in V$ such that $Lg = \langle f, g \rangle$ for all $g \in V$ $\square$

(6) This is essentially the proof of uniqueness of conditional expectation.

Let $A = \{X > Y\}$. Clearly $A \in \mathcal{G}$, so by assumption

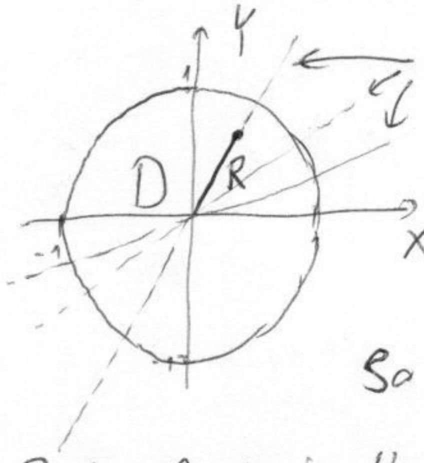
$$\int_A (X-Y)\, d\mathbb{P} = \int_A X\, d\mathbb{P} - \int_A Y\, d\mathbb{P} = 0.$$ Sine $X-Y > 0$ on A,

this means $\mathbb{P}(A) = 0$.

Similarly $\mathbb{P}(Y > X) = 0$, so $\mathbb{P}(X \neq Y) = 0$ $\square$

[Remark: Here $(\Omega, \mathcal{G}, \mathbb{P})$ could be any measure space, $\mathbb{P}(\Omega) = 1$ is not needed.]

(7) Let $R=\sqrt{X^2+Y^2}$, $T=\frac{Y}{X}$ (T is defined with probability 1):



level lines of T

Clearly (by rotation symmetry) R and T are independent,

So $\mathbb{E}(R^2|T)=\mathbb{E}(R^2)$

But if μ is the (joint) distribution of (X,Y), then

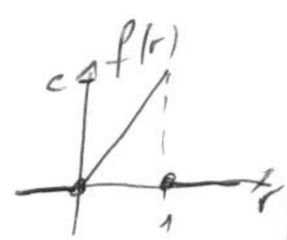
$$\mathbb{E}(R^2)=\mathbb{E}(X^2+Y^2)=\int_{\mathbb{R}^2}(x^2+y^2)\,d\mu(x,y)=\iint_D \frac{1}{\pi}(x^2+y^2)\,dx\,dy$$

$$\underset{\text{polar coordinate}}{=}\int_0^1\int_0^{2\pi}\frac{1}{\pi}r^2 r\,d\varphi\,dr=\int_0^1\frac{2\pi}{\pi}r^3\,dr=2\left[\frac{r^4}{4}\right]_0^1=\frac{1}{2}$$

substitution
$x=r\cos\varphi$
$y=r\sin\varphi$
$dx\,dy=r\,dr\,d\varphi$

[Alternatively: We can see from the geometry that the density of R at r is proportional to the circum-ference of the circle with radius r, so R has density

$$f(r)=\begin{cases}\text{const}\cdot r & \text{if } 0<r<1\\ 0 & \text{if not}\end{cases}$$



Then of course const $=2$ and

$$\mathbb{E}(R^2)=\int_0^1 r^2 f(r)\,dr=\int_0^1 r^2\cdot 2r\,dr=\int_0^1 2r^3\,dr=\frac{1}{2}.$$]

[Remark: Actually R^2 is uniform on $[0,1]$.]

(8) If $F: \mathbb{R} \to [0,1]$ is a distribution function, we define its generalized inverse as $F^{-1}: [0,1] \to \mathbb{R}$ given by

$$F^{-1}(y) := \sup\{x \in \mathbb{R} \mid F(x) < y\}.$$

Then if $Y \sim \mathrm{Uni}[0,1]$, then $F^{-1}(Y)$ has exactly distribution function F.

Now let $\Omega = [0,1]$, let $\mathbb{P} = \mathrm{Leb}$.

Let $F_n(x) = \mu_n((-\infty, x])$ be the distribution fn of μ_n

$F(x) = \mu((-\infty, x])$ ~ μ

Define $X_n, X: \Omega \to \mathbb{R}$ as

$$X_n(w) := F_n^{-1}(w), \quad X(w) = F^{-1}(w).$$

This does the job.