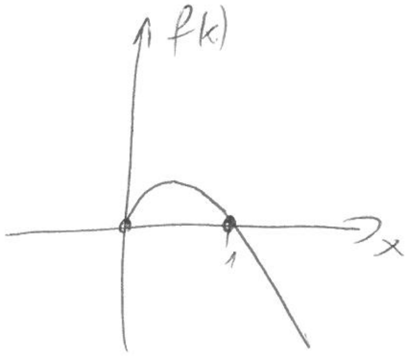
1

$$A_n = \int_0^1 x^{-nx}\,dx = \int_0^1 e^{-nx\ln x}\,dx = \int_0^1 e^{nf(x)}\,dx$$

where
$$f(x) = -x\ln x$$
$$f'(x) = -\ln x - 1$$
$$f''(x) = -\frac{1}{x}$$



The maximum of f is at $0 = f'(x) = -\ln x - 1$

so $\boxed{x = e^{-1}}$

the 2nd derivative at the maximum is

$$-B := f''(e^{-1}) = -\frac{1}{e^{-1}} = -e < 0,$$ so it is indeed a maximum

The value at the maximum is

$$A := f(e^{-1}) = -e^{-1}\underbrace{\ln(e^{-1})}_{-1} = e^{-1}$$

f is twice differentiable* on $[0,1]$ with a unique global maximum at $x_0 = e^{-1}$, so by the ~~B~~ almost Gaussian approximation theorem

$$I_n = \int_0^1 e^{nf(x)}\,dx \sim e^{nA}\sqrt{\frac{2\pi}{nB}} = e^{n/e}\sqrt{\frac{2\pi}{ne}}$$

[* Remark: Actually f is not differentiable at 0, only on $(0,1]$, but this is not a problem – see the remarks to the theorem.]

$\boxed{2}$ Let $Y = X_1 X_2 \cdots X_{10}$. Since μ is the distribution of Y,

$$\int x^3 \, d\mu(x) = \mathbb{E}(Y^3) = \mathbb{E}(X_1^3 X_2^3 \cdots X_{10}^3) \underset{\text{independent}}{\overset{X_1, \dots X_{10} \text{ are}}{=\!=\!=}}$$

$$= E(X_1^3)\, E(X_2^3) \cdots E(X_{10}^3) \overset{X_i \sim Uni[0,1]}{=\!=\!=}$$

$$= \left[\int_0^1 x^3 \, dx\right]^{10} = \left(\left[\frac{x^4}{4}\right]_0^1\right)^{10} = \left(\frac{1}{4}\right)^{10} = \underline{\underline{\frac{1}{2^{20}}}}$$

$\boxed{3}$ $d\nu = g \, d\mu$, so

$$\nu([0,1]) = \int_0^1 1 \, d\nu = \int_0^1 1 \cdot g(y) \, d\mu(y) \overset{\mu = f_* \lambda}{=\!=\!=} \int_{f^{-1}([0,1])} g \circ f \, d\lambda =$$

use the density

integral substitution theorem

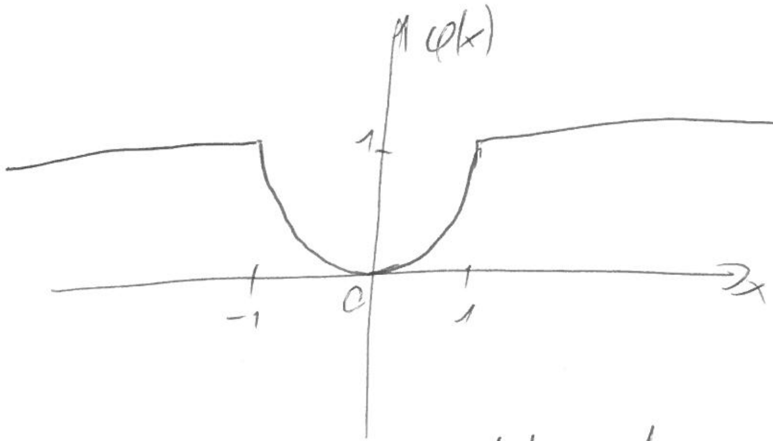
but $f^{-1}([0,1]) = [0,1]$

$$= \int_0^1 (\sqrt{x})^2 \, dx = \int_0^1 x \, dx = \underline{\underline{\frac{1}{2}}}$$

Proof:

[4] Since X_n and X are all bounded by 1,

we have $X_n^2 = \varphi(X_n)$ and $X^2 = \varphi(X)$ where



$$\varphi(x) = \begin{cases} x^2, & \text{if } |x| \leq 1 \\ 1, & \text{if } |x| > 1 \end{cases}$$

So φ is bounded and continuous

$\Rightarrow$ the definition of weak convergence implies

$$\begin{array}{ccc} \mathbb{E}\varphi(X_n) & \xrightarrow{n\to\infty} & E\varphi(X) \\ \| & & \| \\ E(X_n^2) & & E(X^2) \end{array} \quad \square$$

[5] $f_k \geq 0$, so the sequence S_n is monotone increasing

and non-negative and $S_n \to S$ by the definition of

the infinite sum, so the monotone convergence

theorem implies $\int_{\mathbb{R}} S_n d\lambda \xrightarrow{n\to\infty} \int_{\mathbb{R}} S d\lambda$ $\square$

6

a.) Y is a function of X, so it is $\sigma(X)$-measurable

$$\Rightarrow \mathbb{E}(Y|X) = \mathbb{E}(Y|\sigma(X)) = \underline{\underline{Y}}$$

b.) Y only takes 2 values, so $\sigma(Y)$ consists of only 2 atoms:

$$\mathcal{G} := \sigma(Y) = \{\emptyset, \{Y=0\}, \{Y=1\}, \Omega\} = \{\emptyset, \underbrace{[0,1]}_{A}, \underbrace{(1,\infty)}_{B}, [0,\infty)\}$$

A and B have positive probability, so the conditional expectation w.r.t. $\mathcal{G}$ takes the values of the ordinary conditional expectation w.r.t. an <u>event</u>:

$$\mathbb{E}(X|\mathcal{G})\big|_A = \mathbb{E}(X|A) = \frac{\int_A X\,d\mathbb{P}}{\mathbb{P}(A)} = \frac{\int_0^1 w\,e^{-w}\,dw}{\int_0^1 1\cdot e^{-w}\,dw} =$$

$$= \frac{\left[-(w+1)e^{-w}\right]_0^1}{\left[-e^{-w}\right]_0^1} = \frac{1-2e^{-1}}{1-e^{-1}} = \frac{e-2}{e-1}$$

$$\mathbb{E}(X|\mathcal{G})\big|_B = \mathbb{E}(X|B) = \frac{\int_B X\,d\mathbb{P}}{\mathbb{P}(B)} = \frac{\int_1^\infty w\,e^{-w}\,dw}{\int_1^\infty 1\,e^{-w}\,dw} = \frac{\left[-(w+1)e^{-w}\right]_1^\infty}{\left[-e^{-w}\right]_1^\infty} = \frac{2e^{-1}}{e^{-1}} = 2$$

$$\Rightarrow \boxed{\mathbb{E}(X|Y)(w) = \begin{cases} \frac{e-2}{e-1} & \text{if } w \le 1 \\ 2 & \text{if } w > 1 \end{cases}} \quad \boxed{\mathbb{E}(X|Y) = \begin{cases} \frac{e-2}{e-1} & \text{if } Y = 0 \\ 2 & \text{if } Y = 1 \end{cases}}$$

7

$\mathcal{F}_0 = \sigma(\emptyset) = \{\emptyset, \Omega\}$ is the indiscrete σ-algebra,

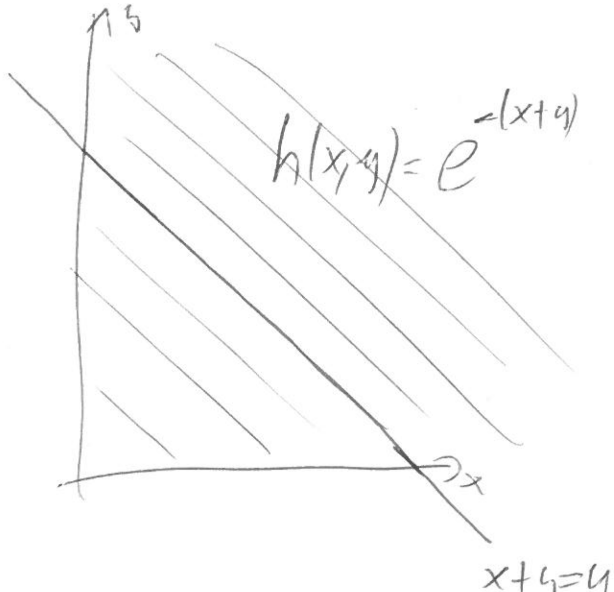
so $\mathbb{E}(Y|\mathcal{F}_0) = \mathbb{E}Y = EX_1 + \dots + EX_4 = 4 \cdot \frac{7}{2} = 14$

is a number, so of course all further conditional expectations leave it unchanged:

$$\text{Answer} = \mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbb{E}(14|\mathcal{F}_1)|\mathcal{F}_2)|\mathcal{F}_3)|\mathcal{F}_4) = \underline{\underline{14}}$$

8

Solution 1: The joint density $h(x,y) := e^{-x} \cdot e^{-y}$ is constant on level lines of $x+y$:



So under the condition $\{X+Y=u\}$, X is uniform on $[0,u]$:

$$E(X^2 | X+Y)\Big|_{X+Y=u} = \int_0^u \frac{1}{u} t^2 \, dt = \frac{1}{u}\left[\frac{t^3}{3}\right]_0^u = \frac{u^2}{3}$$

$$\Rightarrow \mathbb{E}(X^2 | X+Y) = \frac{(X+Y)^2}{3}$$

Solution 2: We have seen in class that $\boxed{U := X+Y}$ and $\boxed{V := \frac{X}{X+Y}}$ are independent and V has a Beta distribution, which happens to be $Uni[0,1]$ in this case. So

$$\boxed{\mathbb{E}(X^2|U)} = \mathbb{E}((VU)^2|U) = \mathbb{E}(U^2V^2|U) \overset{U^2 \in \sigma(U)}{=} U^2 \mathbb{E}(V^2|U) \underset{\text{independence}}{=} U^2 \mathbb{E}V^2 = U^2 \int_0^1 t^2 \, dt = \boxed{\frac{U^2}{3}}$$