Tools of Modern Probability
exam exercise sheet, 25.01.2023
(working time: 90 minutes)

Every exercise is worth 12 points. Every student should *choose* 5 exercises to solve. If more than 5 are solved, I only take the best 5 into account. The maximum total score is 60. Solve easy exercises if you want a good grade. Solve hard ones if you want me to be proud of you.

1. Describe the asymptotic behaviour of the sequence $A_n := \int_{-\pi}^{\pi} 2^{-n\sin^2 x}\,\mathrm{d}x$ as $n \to \infty$. (By "describing the asymptotic behaviour" I mean: find a sequence a_n given by a nice simple formula, such that $A_n \sim a_n$.)

2. Let the random variable X have a uniform distribution on the set $\{1, 2, \dots, 100\}$. Let μ be the distribution of $\sqrt{X}$, $f(x) = \sqrt{x}$ and let $\nu = f_*\mu$. Calculate $\int x^4\,\mathrm{d}\nu(x)$.

3. Let λ be Lebesgue measure on $\mathbb{R}$ and let μ have density $f(x) = e^{-x}$ w.r.t. λ. Let ν be the push-forward of μ by the function $g(x) = x^2$. Calculate $\int_{[0,1]} \sqrt{x}\,\mathrm{d}\nu(x)$.

4. Let $a_{k,l} \in \mathbb{R}$ for every $k, l \in \{0, 1, 2, \dots\}$. Prove the following statements, or give a counterexample:

   a.) $\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{k,l}^2 = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{l,k}^2$
   b.) $\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{k,l}^3 = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{l,k}^3$

   in the sense that if either the left hand side or the right hand side exists then so does the other and they are equal.

5. Let $X, X_1, X_2, \dots \in \mathbb{R}$ be random variables on the same probability space such that $X_{n+1} \geq X_n$ for every n and $X_n \Rightarrow X$. Show that $\mathbb{E}(X_n^2) \to \mathbb{E}(X^2)$, or give a counterexample. Here $\Rightarrow$ denotes weak convergence of random variables.

6. We roll two fair dice and call the results X and Y. Let $\mathcal{G}$ be the σ-algebra generated by $X + Y$. Calculate $\mathbb{E}(X^2|\mathcal{G})$.

7. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space where $\Omega = [0,1] \times [0,1]$, $\mathcal{F}$ is the Borel σ-algebra and $\mathbb{P}(B) = \int_B 3x^2\,\mathrm{d}x\,\mathrm{d}y$ for every Borel set $B \subset \Omega$. Let $U, V : \Omega \to \mathbb{R}$ be $U(x,y) = x$ and $V(x,y) = y$. Calculate $\mathbb{E}(V^2|U)$.

8. Let (X, Y, Z) be jointly Gaussian random variables with $\mathbb{E}X = \mathbb{E}Y = \mathbb{E}Z = 0$, $\mathbb{V}\mathrm{ar}X = \mathbb{V}\mathrm{ar}Y = \mathbb{V}\mathrm{ar}Z = 1$, $\mathrm{Cov}(X,Y) = \frac{1}{3}$ and $\mathrm{Cov}(X,Z) = \mathrm{Cov}(Y,Z) = 0$. Calculate $\mathbb{E}(X|Y+Z)$.