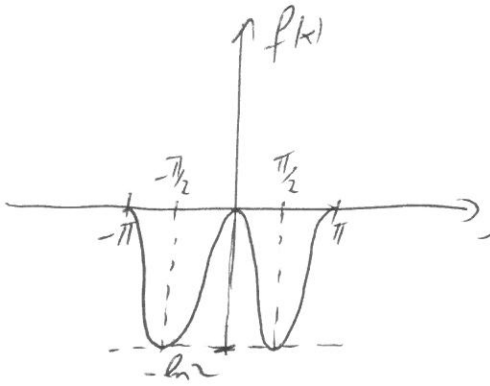
$\boxed{1}$ $A_n = \int_{-\pi}^{\pi} 2^{-n\sin^2 x}\,dx = \int_{-\pi}^{\pi} e^{nf(x)}\,dx$ with $f(x) = -(\ln 2)\sin^2 x$ :




This $f$ does NOT have a unique global maximum on $[a,b] = [-\pi, \pi]$, so the Laplace method for almost Gaussian Integrals does not apply directly. However, $\sin^2 x = \frac{1-\cos(2x)}{2}$ is periodic by $\pi$, so

$$A_n = 2\int_{-\pi/2}^{\pi/2} e^{nf(x)}\,dx$$ with the same $f(x)$.

Now on $[a,b] = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ there is a unique global maximum at $x_0 = 0$ :

$$f(x) = -(\ln 2)\sin^2 x$$
$$f'(x) = -2(\ln 2)\sin x \cos x = -(\ln 2)\cancel{\cos(2x)}\sin(2x)$$
$$f''(x) = -2(\ln 2)\cancel{\sin(2x)}\cos(2x)$$

$\Rightarrow$ $A := f(x_0) = 0$, $f'(x_0) = 0$ as it should be,

$B := -f''(x_0) = 2\ln 2$.

$f$ is also twice differentiable on $[a,b] = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so by the almost Gaussian approximation theorem

$$A_n = 2\int_a^b e^{nf(x)}\,dx \sim 2e^{nA}\sqrt{\frac{2\pi}{nB}} = \underline{\underline{2\sqrt{\frac{\cancel{2}\pi}{n\cancel{2}\ln 2}}}}$$

[2] If the probability space is $(\Omega, \mathcal{F}, \mathbb{P})$ and $Y=\sqrt{X}$, then

$\mu = Y_* \mathbb{P}$, so $\nu = f_* Y_* \mathbb{P} = (f \circ Y)_* \mathbb{P}$.

With $g(x)=x^4$, the integral substitution theorem says

$$\int x^4 \, d\nu(x) = \int_{\mathbb{R}} g \, d\nu = \int_{\Omega} g \circ (f \circ Y) \, d\mathbb{P} = \int_{\Omega} \sqrt{\sqrt{X}}^{\,4} \, d\mathbb{P} =$$

$$= \int_{\Omega} X \, d\mathbb{P} = \mathbb{E}X = \frac{1+2+\dots+100}{100} = \underline{\underline{50.5}}$$

[3] With $h(x)=\sqrt{x}$, the integral substitution theorem says that

$$\int_{[0,1]} \sqrt{x} \, d\nu(x) = \int_{[0,1]} h \, d(g_* \mu) = \int_{g^{-1}([0,1])} h \circ g \, d\mu \; \overset{g^{-1}([0,1])=[-1,1]}{=\!=}$$

~~$h \circ g = tot$~~ $h(g(x)) = \sqrt{x^2} = |x|$

$$= \int_{[-1,1]} |x| \, d\mu(x) \overset{d\mu(x) = e^{-x} d\lambda(x)}{=\!=} \int_{[-1,1]} |x| e^{-x} \, d\lambda(x) \overset{\lambda = \mathrm{Leb}}{=\!=} \int_{-1}^{1} |x| e^{-x} \, dx =$$

$$= \int_{-1}^{0} -x e^{-x} \, dx + \int_{0}^{1} x e^{-x} \, dx = \left[x e^{-x} + e^{-x}\right]_{-1}^{0} - \left[x e^{-x} + e^{-x}\right]_{0}^{1} =$$

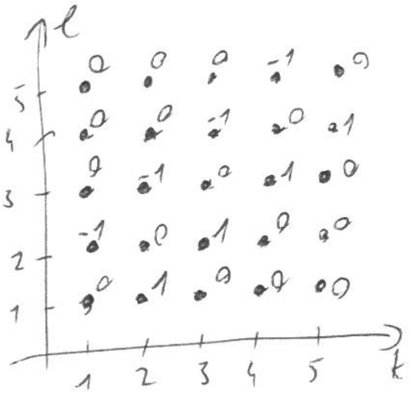
$$= \left\{1 - (\cancel{-e+e})\right\} - \left\{\left(\frac{1}{e} + \frac{1}{e}\right) - 1\right\} = \underline{\underline{2 - \frac{2}{e}}} \approx 1.2642$$

$\boxed{4}$ a.) This is TRUE by the (Fubini)-Tonelli theorem, since $a_{k,l}^2 \geq 0$ always, so the sums (which are integrals w.r.t. counting measure) can be exchanged:

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{k,l}^2 = \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} a_{k,l}^2 \underset{k \leftrightarrow l}{\overset{\text{rename}}{=\!=}} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{l,k}^2$$

b) This is NOT true in general: for example

let $a_{k,l} = \begin{cases} 1, & \text{if } k = l+1 \\ -1, & \text{if } l = k+1 \\ 0 & \text{otherwise} \end{cases}$




Then $a_{k,l}^3 = a_{k,l}$; $a_{l,k} = -a_{k,l}$

$$\sum_{l=1}^{\infty} a_{k,l}^3 = \sum_{l=1}^{\infty} a_{k,l} = \begin{cases} -1, & \text{if } k=1 \\ 0, & \text{if not} \end{cases}$$

$$\Rightarrow \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{k,l}^3 = -1 \quad \text{~~and similarly~~}$$

$$\xRightarrow[a_{k,l}^3 = -a_{l,k}^3]{} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{l,k}^3 = +1$$

✓

[5] **NOT** true: let $Y$ be any non-negative random variable with $\mathbb{E}Y^2 = \infty$, let $\boxed{X_n = -\frac{1}{n} Y}$ and $X \equiv 0$.

Then • $X_n$ is monotone increasing in $n$

• $X_n \to X$ strongly so $X_n \Rightarrow X$ as well

• but $\mathbb{E}(X_n^2) = -\infty$ for every $n$

and $\mathbb{E}(X^2) = 0$, so $\mathbb{E}(X_n^2) \not\to \mathbb{E}(X^2)$

Remark: The statement is true if we assume $X_n \geq 0$. Then $X_\infty := \lim_{n\to\infty} X_n$ exists surely by monotonicity, and the monotone convergence theorem ensures $\mathbb{E}(X_n^2) \longrightarrow \mathbb{E}(X_\infty^2)$.

Moreover, $X_n \to X_\infty$ strongly implies $X_n \Rightarrow X_\infty$, so $X_\infty \sim X$, so $\mathbb{E}(X_\infty^2) = \mathbb{E}(X^2)$

6

$x+y=2$ $\quad$ $x+y=4$ $\quad$ $x+y=6$

$$\Omega = \left\{ \begin{array}{cccccc} 11 & 12 & 13 & 14 & 15 & 16 \\ 21 & 22 & 23 & 24 & 25 & 26 \\ 31 & 32 & 33 & 34 & 35 & 36 \\ 41 & 42 & 43 & 44 & 45 & 46 \\ 51 & 52 & 53 & 54 & 55 & 56 \\ 61 & 62 & 63 & 64 & 65 & 66 \end{array} \right\}$$

$x+y=8$ $\quad$ $x+y=3$ $\quad$ $x+y=10$ $\quad$ $x+y=5$ $\quad$ $x+y=12$

$x+y=7$ $\quad$ $x+y=9$ $\quad$ $x+y=11$

It's easy to see the conditional distribution of $X$ under the condition $\{X+Y=k\}$ (which has positive probability if $k \in \{2,3,4,\dots,12\}$):

| $k$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| conditional distribution of $X$ assuming $X+Y=k$ | $X \equiv 1$ | $Uni(\{1,2\})$ | $Uni(\{1,2,3\})$ | $Uni(\{1,2,3,4\})$ | $Uni(\{1,\dots,5\})$ |

| --- | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|
| | $Uni(\{1,\dots,6\})$ | $Uni(\{2,\dots,6\})$ | $Uni(\{3,\dots,6\})$ | $Uni(\{4,5,6\})$ | $Uni(\{5,6\})$ | $X \equiv 6$ |

$\Rightarrow$

| $k$ | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| $\mathbb{E}(X^2 \mid X+Y=k)$ | $1^2$ | $\frac{1^2+2^2}{2}$ | $\frac{1^2+2^2+3^2}{3}$ | $\frac{1^2+2^2+3^2+4^2}{4}$ | $\frac{1^2+2^2+3^2+4^2+5^2}{5}$ | $\frac{1^2+\dots+6^2}{6}$ |

$\parallel$

$\mathbb{E}(X^2 \mid \mathcal{G})$ on the set $\{X+Y=k\}$ ---

| 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|
| $\frac{2^2+\dots+6^2}{5}$ | $\frac{3^2+4^2+5^2+6^2}{4}$ | $\frac{4^2+5^2+6^2}{3}$ | $\frac{5^2+6^2}{2}$ | $6^2$ |

[6] continued

For those, who like formulas:

$$\mathbb{E}(X^2|\mathcal{G}) = \begin{cases} \dfrac{\sum_{\ell=1}^{X+Y-1} \ell^2}{X+Y-1}, & \text{if } 2 \le X+Y \le 7 \\[2ex] \dfrac{\sum_{\ell=X+Y-6}^{6} \ell^2}{13-(X+Y)}, & \text{if } 7 \le X+Y \le 12 \end{cases}$$

$$= \begin{cases} \dfrac{\cancel{(X+Y-1)}(X+Y)(2(X+Y)-1)}{6\,\cancel{(X+Y-1)}}, & \text{if } 2 \le X+Y \le 7 \\[2ex] \dfrac{91-(X+Y-7)(X+Y-6)\overset{(2(X+Y)-13)}{\cancel{(X+Y-13)}}}{6\,(13-(X+Y))}, & \text{if } 7 \le X+Y \le 12 \end{cases}$$

[7] U and V have joint density $g(x,y) = 3x^2$ on $[0,1]\times[0,1]$

$\Rightarrow$ U has density $g_1(x) = 3x^2$ ~~and~~ on $[0,1]$

V has density $g_2(y) = 1$ on $[0,1]$ : $V \sim \mathrm{Uni}([0,1])$

and they are independent

$$\Rightarrow \mathbb{E}(V^2|U) = \mathbb{E}(V^2) = \int_0^1 y^2\,dy = \frac{1}{3}$$

8 Let $U = Y + Z$, so

$$\mathrm{Cov}(X,U) = \mathrm{Cov}(X, Y+Z) = \mathrm{Cov}(X,Y) + \mathrm{Cov}(X,Z) = \frac{1}{3} + 0 = \frac{1}{3}.$$

~~Then $\mathrm{Cov}(U, X-\lambda U) = \mathrm{Cov}(U,X) - \lambda$~~

and $\mathrm{Cov}(U,U) = \mathrm{Cov}(Y+Z, Y+Z) = \mathrm{Var}\,Y + 2\,\mathrm{Cov}(Y,Z) + \mathrm{Var}\,Z =$

$$= 1 + 0 + 1 = 2.$$

Then for any $\lambda \in \mathbb{R}$

$$\mathrm{Cov}(U, X-\lambda U) = \mathrm{Cov}(U,X) - \lambda\,\mathrm{Cov}(U,U) = \frac{1}{3} - \lambda \cdot 2.$$

So if we choose $\underline{\underline{\lambda = \frac{1}{6}}}$, then $\mathrm{Cov}(U, X-\lambda U) = \mathrm{Cov}(U, X - \frac{U}{6}) = 0$

and they are <u>jointly Gaussian</u> $\Longrightarrow$ INDEPENDENT

$$\Rightarrow \boxed{\mathbb{E}(X \mid U)} = \mathbb{E}\left(X - \frac{U}{6} + \frac{U}{6} \,\middle|\, U\right) = \underbrace{\mathbb{E}\left(X - \frac{U}{6} \,\middle|\, U\right)}_{\overset{\text{independence}}{=} \ \mathbb{E}\left(X - \frac{U}{6}\right)} + \underbrace{\mathbb{E}\left(\frac{U}{6} \,\middle|\, U\right)}_{\overset{\frac{U}{6} \in \sigma(U)}{=} \ \frac{U}{6}} =$$

$$= \mathbb{E}\left(X - \frac{U}{6}\right) + \frac{U}{6} = \underbrace{\mathbb{E}X}_{0} - \frac{1}{6}\underbrace{\mathbb{E}U}_{0} + \frac{U}{6} = \underline{\underline{\frac{U}{6}}}$$