Tools of Modern Probability
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sample exam exercise sheet, fall 2019
(working time: 90 minutes)

1. Show that if X_n, X are real valued random variables and $X_n \to X$ almost surely, then $X_n \Rightarrow X$ (weakly).

2. Construct a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a random variable $X : \Omega \to \mathbb{R}$ and a sequence of random variables $X_n : \Omega \to \mathbb{R}$ such that X_n has uniform distribution on the (finite) set $\left\{0, \frac{2}{n}, \frac{4}{n}, \dots, 2 - \frac{2}{n}, 2\right\}$ and $X_n \to X$ almost surely.

3. Let E be a nonempty, closed, convex set and x a point in a Hilbert space. Show that there is an $e \in E$ such that $d(x, E) = d(x, e)$.

4. Show that if the random variables Y and Z both satisfy the definition of $\mathbb{E}(X|\mathcal{G})$, then $\mathbb{P}(Y = Z) = 1$.

5. Let $\Omega = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$ and let $\mathbb{P} = \frac{1}{36}\chi$ where χ is counting measure on Ω. Let $X : \Omega \to \mathbb{R}$ be defined as $X((i, j)) = i + j$. Describe the measure $X_*\mathbb{P}$.

6. Which of the spaces V below are linear spaces and why?

   a.) $V := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1 + 2x_2 = 0\}$, with the usual addition and the usual multiplication by a scalar.

   b.) $V := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1 + 2x_2 = 3\}$, with the usual addition and the usual multiplication by a scalar.

   c.) $V := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1 \geq 0\}$, with the usual addition and the usual multiplication by a scalar.

   d.) $V := \{f : (0, 1) \to \mathbb{R} \,|\, f \text{ is continuous and } |f| \leq 100\}$, with the usual addition and the usual multiplication by a scalar.

   e.) $V := \{f : (0, 1) \to \mathbb{R} \,|\, f \text{ is continuous and bounded}\}$, with the usual addition and the usual multiplication by a scalar.

7. Let $(X, \mathcal{F})$ be a measurable space and let μ, ν be σ-finite measures on it. Show that there is a countable partition $X = \dot{\bigcup}_i A_i$ such that $\mu(A_i) < \infty$ and $\nu(A_i) < \infty$ for every i. Use this to show that the special case of the Radon-Nikodym theorem for finite measures implies the general theorem (for σ-finite measures).

8. Let U and V be independent random variables, uniformly distributed on $[0, 1]$. Calculate $\mathbb{E}(\sqrt{U + V}|U)$.