Tools of Modern Probability
sample exam exercise sheet, fall semester 2022-23
(working time: 90 minutes)

Every exercise is worth 12 points. Every student should *choose* 5 exercises to solve. If more than 5 are solved, I only take the best 5 into account. The maximum total score is 60. Solve easy exercises if you want a good grade. Solve hard ones if you want me to be proud of you.

1. Describe the asymptotic behaviour of the following sequences as $n \to \infty$:

   a.) $A_n := \int_0^{2\pi} \cos^{2n} x \,\mathrm{d}x$

   b.) $B_n := \int_{-1}^{1} (x-1)^n (x+1)^n \,\mathrm{d}x$

   (By "describing the asymptotic behaviour" I mean: find sequences a_n and b_n given by nice simple formulas, such that $A_n \sim a_n$ and $B_n \sim b_n$.)

2. For $d = 1, 2, 3, \ldots$ let c_d be the surface volume of the d-dimensional unit sphere

$$S_d := \left\{ x \in \mathbb{R}^d \,\middle|\, |x| = 1 \right\}.$$

   Show that $c_d = 2\frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$.

3. Let $\mathbb{P}$ be Lebesgue measure on $[0,1] \times [0,1]$, let $X : [0,1] \times [0,1] \to \mathbb{R}$ be given by $X(u,v) = \sqrt{u+v}$ and let $\mu = X_* \mathbb{P}$. Calculate $\int_{\mathbb{R}} x^2 \,\mathrm{d}\mu(x)$.

4. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f, f_1, f_2, f_3, \cdots : X \to \mathbb{R}$ be measurable functions such that $f_n(x) \to f(x)$ for μ-almost every $x \in X$ and $|f_n(x)| \le 1$ for all x and n. In the following two special cases, show that

$$\int_X \lim_{n\to\infty} f_n(x) \,\mathrm{d}\mu(x) \le \lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu,$$

   or give a counterexample:

   a.) $X = \mathbb{R}$, $\mu = Leb$

   b.) $X = [0,1]$ and μ is a probability measure on X.

5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $X : \Omega \to \mathbb{R}$ integrable. Let $\mathcal{G}_1 \subset \mathcal{F}$ and $\mathcal{G}_2 \subset \mathcal{F}$ be sub-σ-algebras. Show that

$$\mathbb{E}(\mathbb{E}(X \,|\, \mathcal{G}_1) \,|\, \mathcal{G}_2) = \mathbb{E}(\mathbb{E}(X \,|\, \mathcal{G}_2) \,|\, \mathcal{G}_1),$$

   or give a counterexample.

6. Let X and Y be independent, uniformly distributed on $[-1,1]$. Calculate $\mathbb{E}(X \,|\, X+Y)$.

7. Let X be uniformly distributed on $[-1,2]$. Calculate $\mathbb{E}(X|X^2)$.

8. Let $X, X_1, X_2, X_3, \ldots$ be real valued random variables and let $F, F_1, F_2, F_3, \ldots$ be their distribution functions. Show that $X_n \Rightarrow X$ if and only if $F_n \Rightarrow F$. Here $\Rightarrow$ denotes weak convergence. (*Show* means: sketch the proof.)