Budapest University of Technology and Economics
Institute of Mathematics

Moments and Cumulants in Renewal Processes

Bachelor Thesis

Author:
Ádám Bernát

Consultant:
Péter Bálint

2022

Contents

1 The Model

1.1 Motivation

In this thesis we study a random walk generated by a renewal process with polynomial tail distribution. The motivation comes from the study of billiards, which are chaotic systems. This is an actively researched topic (see the monograph [6] for detail). The main idea is that a billiard ball is initialized from some point with a given direction. There are scattering objects on $\mathbb{R}^2$, placed in a periodic manner, which, if hit by the ball, will reflect it into a direction, with the classic angle of incidence = angle of reflection law. If the scatterers are convex, then from the chaotic manner of this system, after a handful of reflections the position of the ball can be thought of as random, independent of the starting point.

A system has the "infinite horizon" property if the possible travel lengths of a billiard ball, in between hitting scatterers is not bounded from above. One model of great interest, which may possess the infinite horizon property, is the Lorentz model where a gas molecule is "playing the role of the billiard ball", bouncing around when hitting the scatterers. This model and its properties are discussed in great detail in [8].

A simple case is when circular scattering objects are placed on the lattice points of $\mathbb{Z}^2$ with radius small enough that they do not intersect. The non-intersecting placement ensures the infinite horizon property, because a molecule can wonder into a "corridor" where theoretically it could fly for an arbitrary long time.

We are going to present a simpler model that is inspired by the analogy: There is a particle, which moves on a 1 dimensional line and it randomly changes direction, which can be interpreted as collisions. The consecutive directions are ± 1 with probabilities $\frac{1}{2} - \frac{1}{2}$, the intercollision times are also i.i.d. and thus the collision moments give us a renewal process. Here in our setup arrival distributions have polynomial tails.

Obviously this is a huge simplification for a particle movement, but it still captures the wandering in a corridor where exact vertical placement is almost negligible, all the important data are the hitting times and horizontal travel directions. The true billiard process is more complicated though, long flights in corridors interchange with a series of shorter flights, which, as a first approximation can be regarded as the random change of direction of the corridor motion.

The infinite horizon property implies a tail distribution. The already studied case is $\mathbb{P}(L > x) \sim x^{-2}$ (see [5] for detail). As a consequence, such Lorentz models have limit theorems different from that of the standard Central Limit Theorem. In particular, although, the limit remains Gaussian, a $\sqrt{n \ln(n)}$ scaling is needed instead of the standard $\sqrt{n}$, thanks to the infinite horizon property. These non-standard limit laws are described in [9].

Furthermore, for this special case of tail distribution $\sim x^{-2}$ a discrepancy between convergence in distribution and convergence of moments was observed ([5], [7], [10]), also discussed later here in subsection 1.4. This only gets more interesting by the observation made in [5] appendix A, which introduces the same model that we study in this present thesis (subsection 1.2). This shows that the discrepancy between convergence in distributions and second moments is not a result of the physical model or some effect of the chaotic system, but it is a mere consequence of the tail $\sim x^{-2}$, as this pure probabilistic model also yields these "paradoxes".

We intend to further investigate this kind of behavior by extending the model of [5], Appendix A to other tail distributions and studying not only the variances, but also higher cumulants of the tail distribution. A similar approach can be found in [7], which also studies the behavior of different cumulants in a billiard model with tail distribution $\sim x^4$.

Finally, let us note that distributions with polynomial decay occur in many areas of mathematics and applications, so there is a chance that results obtained here could be applicable elsewhere.

1.2 Description

Our model has the following setup. Let us consider a renewal process with arrival times $0 < T_0 < T_1 < T_2 < \dots$. Let $\xi(x)$ be the process, which has values 1 or -1 in between arrival times, with even probabilities (That is $+1/-1$ with probability $\frac{1}{2}$)

More precisely let ξ_i $\forall i \in \mathbb{N}$ be i.i.d. random variables with probabilities: $\mathbb{P}(\xi_i = 0) = \mathbb{P}(\xi_i = 1) = \frac{1}{2}$. Now $\xi(x)$ is the process defined as $\xi(x) := \xi_i$ if $T_{i-1} < x < T_i$ $i \geq 1$ and $\xi(x) := \xi_0$ if $x < T_0$

Moreover, for some $a > 0$ and $\beta \geq 2$ fixed, let $L_k = T_k - T_{k-1}$ be independent identically distributed random variables (the k'th interarrival time) with polynomial tail behavior $\mathbb{P}(L > x) \sim ax^{-\beta}$, but occasionally we are going to assume an ever slightly stronger condition, that is $\mathbb{P}(L > x) = ax^{-\beta} + \mathcal{O}(x^{-\beta-1})$ as $x \to \infty$ $(a > 0, \beta \geq 2)$ and let the L_k's have distribution L. Also let $\mu = \mathbb{E}(L)$

Here, and throughout my thesis , $a(t) \sim b(t)$ means that :

$$\frac{a(t)}{b(t)} \to 1$$

as $t \to \infty$

Now let us define the two main subjects of our investigation:

$$S_T := \int_0^T \xi(t)dt \tag{1}$$

and

$$X_k := \xi_k L_k, \quad S_n = \sum_{i=1}^{n} X_k \tag{2}$$

Where $k \geq 1$ and $n \geq 1$.
Also X_k are all i.i.d. random variables, so let's call their common distribution X. Note that $|X| = L$, yet the distribution of X is symmetric so $\mathbb{E}(X) = 0$ and $\mathbb{E}(X^3) = 0$
These two quantities are related. If $T = T_n$ is an arrival time for some $n \in \mathbb{N}$ then

$$(S_{T_n} =)S_T = S_n + \xi_0 T_0$$

For each $t > 0$ let's define: $m(t) = min\{m \in \mathbb{N} : t < T_m\}$ and then
$H(t) = m(t) - t$ the residual time. To ensure stationarity let T_0 have distribution:

$$\mathbb{P}(H(t) > u) = \mathbb{P}(T_0 > u) \tag{3}$$

Also it follows from [2] Chapter 3 (Theorem 3.9) , (because of the sized biased distribution) , that:

$$\mathbb{P}(T_0 > t) = \frac{1}{\mu}\int_t^\infty \mathbb{P}(L > x)dx$$

1.3 Moments and Cumulants

We are going to define the cumulants just like in [1] Chapter I §2

Definition 1.1. *Let X be a random variable. It's logarithmic characteristic function is defined as $ln(\psi_X(t)) = \ln(\mathbb{E}(e^{itX}))$. If $ln(\psi_X(t))$ is n times differentiable, then the n'th cumulant of X is:*

$$\mathcal{C}_n(X) := \frac{1}{i^n}\left[\frac{d^n}{dt^n}\ln(\mathbb{E}(e^{itX}))\right]_{t=0}$$

It is easy to see that if the logarithmic characteristic function has a Taylor expansion around 0 then it is:

$$\ln(\mathbb{E}(e^{itX})) = \sum_{k=1}^{\infty}\frac{\mathcal{C}_k(X)}{k!}(it)^k \tag{4}$$

So the cumulants are easily obtainable from the Taylor expansion.
Some special cumulants are $\mathbb{E}(X) = \mathcal{C}_1(X)$ and $Var(X) = \mathcal{C}_2(X)$. Higher cumulants do not coincide with central moments but can be expressed in terms of them. In particular we have the following property:

Lemma 1.2. *Assume X is a random variable, whose 4th moment exists and $\mathbb{E}(X) = 0$, then the 4th cummulant of X can be expressed as:*

$$\mathcal{C}_4(X) = \mathbb{E}(X^4) - 3(\mathbb{E}(X^2))^2$$

Proof. See [1] Chapter I. □

Here it is worth noting that if X has standard normal distribution, then :

$$\ln(\mathbb{E}(e^{itX})) = -\frac{1}{2}t^2$$

So all cumulants (except the second , which is the variance) are 0. This is why cumulants, in some way, measure the "distance" from the standard normal distribution.

1.4 Our Goal

Now that the model is clear, let's ask the question: why are we doing this ? The original idea comes from [5] (Appendix A).

Here the case $\beta = 2$ is discussed , where the Central Limit Theorem is not applicable, since the variances don't exist. The observation made there was, that using a modified Central limit theorem on the i.i.d. sum S_n (2) it is true that (by [3] Section XVII.5, Theorem 2):

$$\frac{S_n}{\sqrt{n \cdot \ln(n)}} \to \mathcal{N}(0, a)$$

Here 'a' is the constant of the L distribution. From this Central Limit Theorem used for the i.i.d. sum S_n,we can conclude by the Law of Large Numbers that the renewal process also converges in distribution:

$$\frac{S_T}{\sqrt{T \ln(T)}} \to \mathcal{N}\left(0, \frac{a}{\mu}\right)$$

In contrast to the limit of the second moment calculated in Appendix A of [5]:

$$\lim_{n\to\infty} \mathbb{E}\left(\frac{S_T^2}{T\ln(T)}\right) \to \frac{2a}{\mu}$$

Here this doubling effect is a direct analogue of what has been observed in infinite horizon Lorentz models ([7],[10]) (The μ is there because of the renewal process, but there we would expect to find $\frac{a}{\mu}$). We would like to further investigate the extent of these interesting phenomena, by asking the question: what happens to different cumulants at the relevant 'critical tail distribution'. In particular we focus on the behavior of the 4th cumulant at $\beta \geq 4$. First calculating it for S_n in the I.I.D. sums section. Later in the Renewal Process part calculating for S_T. However, in contrast to $\beta = 2$, here we are going to have the classical Central Limit Theorem , since the second moment of X exists.

2 I.I.D. Sums

Lemma 2.1. *Let X, Y be independent random variables and let $n \in \mathbb{N}$.*

$$\mathcal{C}_n(X+Y) = \mathcal{C}_n(X) + \mathcal{C}_n(Y)$$

Proof. It is trivial with the properties of the characteristic function, that:

$$\ln(\mathbb{E}(e^{it(X+Y)})) = \ln(\mathbb{E}(e^{itX})) + \ln(\mathbb{E}(e^{itY})) =$$

Rewriting this with the help of (4) :

$$\sum_{k=1}^{\infty}\frac{\mathcal{C}_k(X+Y)}{k!}(it)^k=\sum_{k=1}^{\infty}\frac{\mathcal{C}_k(X)}{k!}(it)^k+\sum_{k=1}^{\infty}\frac{\mathcal{C}_k(Y)}{k!}(it)^k=\sum_{k=1}^{\infty}\frac{\mathcal{C}_k(X)+\mathcal{C}_k(Y)}{k!}(it)^k$$

Now using the fact that 2 Taylor series are equal if and only if their coefficients are all equal. We get the desired result. □

Here we fix some notation.

Definition 2.2. *Let $f_L(x)$ be the density function of the distribution of L.*
Let $f_X(x)$ be the density function of the distribution of X.
Let $\psi_X(t) := \mathbb{E}(e^{itX})$ denote the characteristic function of X at t.

In the case where $\beta > 4$:

$$f_L(x) = \beta a x^{-(\beta+1)} + o(x^{-(\beta+1)}) \qquad as \quad x \to +\infty$$

$$f_X(x) = f_X(-x) = \frac{\beta}{2} a x^{-(\beta+1)} + o(x^{-(\beta+1)}) \qquad as \quad x \to +\infty$$

Now lets consider S_n defined in (2). According to the Central Limit Theorem: (Because 2nd moment exists)

$$\frac{S_n}{\sqrt{n}} \to \mathcal{N}(0,\sigma^2) \qquad as \;\; n \to \infty$$

(Here $\mathbb{E}(X) = 0$, and $\mathbb{E}(X^2) = Var(X) = \sigma^2$). What we would like to approximate is the "rate" at which it converges to the normal distribution.

It follows easily from the properties of the characteristic function (see [3] chapter XV, lemma 1):

$$\ln(\psi_{S_n/\sqrt{n}}(t)) = \ln([\psi_X(t/\sqrt{n})]^n) = n \cdot \ln([\psi_X(t/\sqrt{n})]) \tag{5}$$

In the $\beta > 4$ case the fourth cumulant exists; also, because of the symmetric nature of X ($\mathcal{C}_1(X) = 0$ and $\mathcal{C}_3(X) = 0$) the Taylor expansion will look like:

$$\ln(\psi_{S_n/\sqrt{n}}(t)) = -\frac{\sigma^2 t^2}{2} + \frac{\mathcal{C}_4(X)}{24}\frac{t^4}{n} + o\left(\frac{1}{n}\right) \qquad as \;\; n \to \infty, \;\; for\ any\ t \in \mathbb{R}$$

In the case where $\beta = 4$:

$$f_L(x) = 4ax^{-5} + o(x^{-5}) \qquad as \quad x \to +\infty$$

$$f_X(x) = f_X(-x) = 2ax^{-5} + o(x^{-5}) \qquad as \quad x \to +\infty$$

Here what we actually have to approximate is $ln(\psi_X(t))$, which by (5) provides directly an expansion of $\ln(\psi_{S_n/\sqrt{n}}(t))$. We will do this by finding its Taylor series. The method is from [4] (proof of proposition 1.1). Which focuses on the cases when $\beta \leq 2$, in particular for $\beta = 2$ gets:

$$ln(\psi_X(t)) = -at^2 \ln\left(\frac{1}{t}\right) + \mathcal{O}(t^2) \quad as \;\; t \to \infty$$

from which, the well-known non-standard limit law follows:

$$ln(\psi_{\frac{S_n}{\sqrt{n\ln(n)}}}(t)) = -\frac{at^2}{2} + o(1) \qquad as \;\; n \to \infty, \;\; for \; any \; t \in \mathbb{R}$$

Since this paper considers only $\beta \leq 2$ we have to modify its methods to work properly in our case ($\beta = 4$).

$$\mathbb{E}(e^{itX}) = \int_{-\infty}^{\infty} e^{itx} f_X(x)dx = M + \int_{1}^{\infty} e^{itx} 2ax^{-5}dx + \int_{-\infty}^{-1} e^{itx} 2ax^{-5}dx \tag{6}$$

Here M denotes:

$$M = \int_{-1}^{1} e^{itx} f_X(x)dx + \int_{1}^{\infty} e^{itX} o(x^{-5})dx + \int_{-\infty}^{-1} e^{itx} o(x^{-5})dx$$

Let us fix $t > 0$. If $t < 0$ similar computation applies with $|t|$ instead of t. So now we need to calculate the integral in (6), let's name it Ω. Here is where we rely on [4], (also introducing $z = tx$) :

$$\Omega_+ := \int_{x=1}^{\infty} e^{itx} 2ax^{-5}dx = 2a \int_{z=t}^{\infty} e^{iz} \left(\frac{z}{t}\right)^{-5} \frac{1}{t}dz = 2at^4 \int_{z=t}^{\infty} e^{iz} z^{-5}dz$$

And with $z = -tx$:

$$\Omega_- := \int_{x=-\infty}^{-1} e^{itx} 2ax^{-5}dx = 2a \int_{z=\infty}^{t} e^{-iz} \left(\frac{-z}{t}\right)^{-5} \frac{1}{t}dz =$$

$$= 2at^4 \int_{z=t}^{\infty} e^{-iz} z^{-5}dz = 2at^4 \overline{\int_{z=t}^{\infty} e^{iz} z^{-5}dz} = \overline{\Omega_+}$$

Definition 2.3. *Let ϕ denote the function:*

$$\phi(z) := 1 + iz - \frac{z^2}{2} - \frac{iz^3}{6} + \frac{z^4}{24}\mathbb{I}\{z \leq 1\}$$

Where $\mathbb{I}\{z \leq 1\}$ is the indicator of the event $\{z \leq 1\}$.

Now we can say that:

$$\Omega_+ = 2at^4 \int_{z=t}^{\infty} e^{iz} z^{-5} dz = 2at^4 \int_{z=t}^{\infty} \Big[e^{iz} - \phi(z)\Big] z^{-5} dz + 2at^4 \int_{z=t}^{\infty} \phi(z) z^{-5} dz \tag{7}$$

Below we show that the first integral will be of $\mathcal{O}(t^4)$, thanks to the carefully designed ϕ. Indeed

The $[t, 1]$ part:

$$2at^4 \int_{z=t}^{1} \Big[e^{iz} - \phi(z)\Big] z^{-5} dz = 2at^4 \int_{z=t}^{1} \left[z^{-5} \sum_{k=5}^{\infty} \frac{(iz)^k}{k!}\right] dz = \mathcal{O}(t^4) \tag{8}$$

The last step stands, as $z^{-5} \sum_{k=5}^{\infty} \frac{(iz)^k}{k!}$ is clearly the Taylor series of a well defined and continuous function on $[0, 1]$, meaning it is bounded there as well. Thus the integral on [t,1] is of $\mathcal{O}(1)$ as $t \to 0$.

The $[1, \infty]$ part:

$$\left|2at^4 \int_{z=1}^{\infty} \Big[e^{iz} - \phi(z)\Big] z^{-5} dz\right| \leq 2at^4 \int_{z=1}^{\infty} z^{-5} \left|e^{iz} - \phi(z)\right| dz \leq$$

$$\leq 2at^4 \int_{z=1}^{\infty} z^{-5} \left|e^{iz} - 1 - iz + \frac{z^2}{2} + \frac{iz^3}{6}\right| dz \leq$$

$$\leq 2at^4 \int_{z=1}^{\infty} \left[\left|\frac{e^{iz}}{z^5}\right| + \left|\frac{1}{z^5}\right| + \left|\frac{i}{z^4}\right| + \left|\frac{1}{2z^3}\right| + \left|\frac{i}{6z^2}\right|\right] dz \leq$$

$$\leq 2at^4 \int_{z=1}^{\infty} \left[\frac{1}{z^5} + \frac{1}{z^5} + \frac{1}{z^4} + \frac{1}{2z^3} + \frac{1}{6z^2}\right] dz = \mathcal{O}(t^4) \tag{9}$$

Where the last step is true, since the integral is finite, meaning it is of $\mathcal{O}(1)$

The second integral in (7):

$$\int_{z=t}^{\infty} \phi(z)z^{-5}dz = \int_{z=t}^{\infty} \left[1 + iz - \frac{z^2}{2} - \frac{iz^3}{6} + \frac{z^4}{24}\mathbb{I}\{z \leq 1\}\right]z^{-5}dz =$$

$$= \int_{z=t}^{\infty} \left[\frac{1}{z^5} + \frac{i}{z^4} - \frac{1}{2z^3} - \frac{i}{6z^2} + \frac{1}{24z^1}\mathbb{I}\{z \leq 1\}\right]dz =$$

$$= \left[\frac{-1}{4z^4} - \frac{i}{3z^3} + \frac{1}{4z^2} + \frac{i}{6z}\right]_t^{\infty} + \left[+\frac{1}{24}ln(z)\right]_t^{1} = \frac{1}{4t^4} + \frac{i}{3t^3} - \frac{1}{4t^2} - \frac{i}{6t} - \frac{1}{24}\ln(t) \quad (10)$$

Some observations regarding these terms. Note that these terms are multiplied by a factor at^4 in (9), thus their powers in t are increased by 4. If we write them back into (6), it follows from the definitions of the cumulants (def 1.1) that:

- $\frac{1}{4t^4}$ together with M and Ω_- will give us a 1, as $\mathbb{E}(e^{itX})$ is a characteristic function.
- $\frac{i}{3t^4}$ together with M and Ω_- will give us a $0 \cdot t$, as the $\mathbb{E}(X) = 0$.
- $\frac{1}{4t^2}$ together with M and Ω_- will give us a $\frac{-\sigma^2 t^2}{2}$, as the Variance of X is σ^2.
- $\frac{1}{6t}$ together with M and Ω_- will give us a $0 \cdot t^3$, as for symmetric distributions $\mathbb{E}(X^3)$ is also 0.
- $\frac{1}{24}\ln(t)$, which corresponds to the term $t^4 \cdot \ln\left(\frac{1}{t}\right)$ will be the same in Ω_-. As $\Omega_- = \overline{\Omega_+}$ and this term has a real coefficient
- $t^4 \cdot \ln\left(\frac{1}{t}\right)$ is missing from M, where the next term after t^3 will be t^4 which $= \mathcal{O}(t^4)$.

Which means that the first relevant term after t^2 is $t^4\ln(t)$. Now combining all of this by using the equations (8),(9),(10) and putting the above mentioned observations about the characteristic function back into (6) we get:

$$\mathbb{E}(e^{itX}) = M + \int_1^{\infty} e^{itx}2ax^{-5}dx + \int_{-\infty}^{-1} e^{itx}2ax^{-5}dx =$$

$$1 + 0t - \frac{\sigma^2 t^2}{2} + 0t^3 + 2 \cdot 2at^4\frac{-1}{24}\ln(t) + \mathcal{O}(t^4) = 1 - \frac{\sigma^2 t^2}{2} + \frac{-a}{6}t^4\ln(t) + \mathcal{O}(t^4) \quad (11)$$

This is all good, however we actually need the logarithmic characteristic function. However this is not a problem, we just need to find the Taylor expansion of $\ln(1+x)$:

$$\ln(1+x) = 0 + x - \frac{x^2}{2} + o(x^2)$$

by substituting $1 + \Big(- \frac{\sigma^2 t^2}{2} + \frac{-a}{6} t^4 \ln(t) + \mathcal{O}(t^4) \Big)$ into the place of $1+x$:

$$\ln([\psi_X(t)]) = \ln(\mathbb{E}(e^{itX})) = \Big(- \frac{\sigma^2 t^2}{2} + \frac{-a}{6} t^4 \ln(t) + \mathcal{O}(t^4) \Big) + \mathcal{O}(t^4) =$$

$$= -\frac{\sigma^2 t^2}{2} + \frac{a}{6} t^4 \ln\left(\frac{1}{t}\right) + \mathcal{O}(t^4) \tag{12}$$

It is because $\Big(- \frac{\sigma^2 t^2}{2} + \frac{-a}{6} t^4 \ln(t) + \mathcal{O}(t^4) \Big)^2$ is already of $\mathcal{O}(t^4)$, as are the other higher terms in $\ln(1+x)$.

<u>Lastly calculating $\ln(\psi_{S_n/\sqrt{n}}(t))$</u>: Let's use (5) and (12)

$$\ln(\psi_{S_n/\sqrt{n}}(t)) = n \cdot \ln([\psi_X(t/\sqrt{n})]) =$$

$$= n \left[-\frac{\sigma^2}{2} \left(\frac{t}{\sqrt{n}} \right)^2 + \frac{a}{6} \left(\frac{t}{\sqrt{n}} \right)^4 \cdot \ln\left(\frac{\sqrt{n}}{t} \right) + \mathcal{O}\left(\left(\frac{t}{\sqrt{n}} \right)^4 \right) \right] =$$

$$= -\frac{\sigma^2 t^2}{2} + \frac{a t^4}{6n} \left[\ln\left(\frac{1}{t}\right) + \frac{1}{2} \ln(n) \right] + \mathcal{O}\left(\frac{t^4}{n} \right)$$

The next step is to look at this expression for fixed t and determine what happens as $n \to \infty$. Now we get what we were looking for by using that if $f(x) = \mathcal{O}(x^4)$ around 0, then $f(x) = o(x^4 \ln(\frac{1}{x}))$. So the dominant term after the constant in n is $\frac{\ln(n)}{n}$:

$$\ln(\psi_{S_n/\sqrt{n}}(t)) = -\frac{\sigma^2 t^2}{2} + \frac{a \cdot t^4}{12} \frac{\ln(n)}{n} + \mathcal{O}\left(\frac{t^4}{n} \right) \tag{13}$$

3 Renewal Process

Our goal in this section is to give estimates on $\mathcal{C}_4(S_T)$ with the help of lemma 1.2. We start with an easy case where L has exponential distribution (Poisson point process). Then we calculate it for a general L with $\beta > 4$, where the calculations are much harder. (Comparing the results to the exponential case we should get the same results.) Then it will be easier to construct the somewhat special case of $\beta = 4$ from the arguments made

in the general ($\beta > 4$) case, which we will do so in the last part of this section.
To compute $\mathcal{C}_4(S_T)$, we need the moments $\mathbb{E}(S_T^4)$ and $\mathbb{E}(S_T^2)$ (according to lemma 1.2). We don't need to calculate everything exactly and we will use different precision for different cases. (The exponential will be calculated exactly)

The case where $\beta > 4$: We will find that $o(T)$ precision will suffice. As the leading terms in $\mathbb{E}(S_T^4)$ and in $3(\mathbb{E}(S_T^2))^2$, both $\sim T^2$, will cancel out; leaving us with a $constant * T + o(T)$ expression for $\mathcal{C}_4(S_T)$. Here the constant will be precisely calculated, but for the other terms we will only show that they are $o(T)$.

The case where $\beta = 4$: Here we can be even less accurate, as the remaining leading term after the cancellation will be of order $T\ln(T)$. Calculating its exact coefficient is necessary; however, the terms with less significance ($\mathcal{O}(T)$) are not needed exactly.

3.1 Notations

Let us introduce a notation $G_n(s)$ which will be useful to us. It is defined recursively in the following way:

Definition 3.1. *Let*

$$G_0(s) = \frac{1}{\mu}\mathbb{P}(L > s)$$

where μ is chosen in a way that $G_0(s)$ is a density function supported on $\mathbb{R}^+$ (The density function of T_0). Now we can define:

$$G_1(s) := \mathbb{P}(T_0 > s) = \frac{1}{\mu}\int_s^\infty \mathbb{P}(L > x)dx \qquad for \quad s \geq 0$$

If $G_n(s)$ is already defined then:

$$G_{n+1}(s) := \int_s^\infty G_n(x)dx \qquad for \quad s \geq 0$$

Note that $G_n(s)$ for $n \geq 1$ is not a density function.

We can come up with some approximations for $G_n(s)$ (assuming β is large enough in the setup described above)

Lemma 3.2. *if $n < \beta$ then $G_n(s)$ exists and:*

$$G_n(s) = \frac{a}{\mu\prod_{i=1}^n(\beta - i)}s^{-(\beta-n)} + \mathcal{O}(s^{-(\beta-n+1)})$$

as $s \to \infty$

Proof. Induction in n. The first step is for n = 1

$$G_1(s) = \mathbb{P}(T_0 > s) = \frac{1}{\mu}\int_s^\infty \mathbb{P}(L_k > x)dx = \frac{1}{\mu}\int_s^\infty ax^{-\beta} + \mathcal{O}(x^{-(\beta+1)})dx =$$
$$= \left[\frac{-a}{\mu(\beta-1)}x^{-(\beta-1)} + \mathcal{O}(x^{-(\beta)})\right]_{x=s}^\infty = \frac{a}{\mu(\beta-1)}s^{-(\beta-1)} + \mathcal{O}(s^{-(\beta)}).$$

Assume true until n-1, then:

$$G_n(s) = \int_s^\infty G_{n-1}(x)dx = \int_s^\infty \frac{a}{\mu\prod_{i=1}^{n-1}(\beta-i)}x^{-(\beta-n+1)} + \mathcal{O}(x^{-(\beta-n+2)})dx =$$
$$= \left[\frac{-a}{(\beta-n)\mu\prod_{i=1}^{n-1}(\beta-i)}x^{-(\beta-n)} + \mathcal{O}(x^{-(\beta-n+1)})\right]_{x=s}^\infty =$$
$$= \frac{a}{\mu\prod_{i=1}^{n}(\beta-i)}s^{-(\beta-n)} + \mathcal{O}(s^{-(\beta-n+1)})$$

□

Another equality, which is not hard to see is the following:

Lemma 3.3. *if $n < \beta$ then*

$$G_n(0) = \frac{1}{(n-1)!}\mathbb{E}(T_0^{n-1}) = \frac{1}{n!\mu}\mathbb{E}(L^n)$$

Where L has the same distribution as the L_k's , also T_0 has density function $G_0(t)$

Proof.

$$G_n(0) = \int_{t_1=0}^\infty G_{n-1}(t_1)dt_1 = \int_{t_1=0}^\infty\int_{t_2=t_1}^\infty G_{n-2}(t_2)dt_2dt_1 =$$
$$= \int_{t_2=0}^\infty\int_{t_1=0}^{t_2} G_{n-2}(t_2)dt_1dt_2 = \int_{t_2=0}^\infty t_2G_{n-2}(t_2)dt_2 = \cdots =$$
$$= \int_{t_{n-1}=0}^\infty \frac{t_{n-1}^{n-2}}{(n-2)!}G_1(t_{n-1})dt_{n-1} = \int_{t_{n-1}=0}^\infty \frac{t_{n-1}^{n-2}}{(n-2)!}\mathbb{P}(T_0 > t_{n-1})dt_{n-1} =$$
$$= \int_{t_{n-1}=0}^\infty\int_{t_n=t_{n-1}}^\infty \frac{t_{n-1}^{n-2}}{(n-2)!}G_0(t_n)dt_ndt_{n-1} = \int_{t_n=0}^\infty\int_{t_{n-1}=0}^{t_n} \frac{t_{n-1}^{n-2}}{(n-2)!}G_0(t_n)dt_{n-1}dt_n =$$
$$= \int_{t_n=0}^\infty \frac{t_n^{n-1}}{(n-1)!}G_0(t_n)dt_n = \frac{1}{(n-1)!}\mathbb{E}(T_0^{n-1})$$

Since $G_0(t)$ is the density function of T_0

The second equation has the same proof save stopping at the last step. L has density function $f_L(t)$

$$G_n(0) = \int\limits_{t_n=0}^{\infty} \frac{t_n^{n-1}}{(n-1)!} G_0(t_n) dt_n = \int\limits_{t_n=0}^{\infty} \frac{t_n^{n-1}}{(n-1)!} \frac{1}{\mu} \mathbb{P}(L > t_n) dt_n =$$

$$= \int\limits_{t_n=0}^{\infty} \int\limits_{t_{n+1}=t_n}^{\infty} \frac{t_n^{n-1}}{(n-1)!} \frac{f_L(t_{n+1})}{\mu} dt_{n+1} dt_n = \int\limits_{t_{n+1}=0}^{\infty} \int\limits_{t_n=0}^{t_{n+1}} \frac{t_n^{n-1}}{(n-1)!} \frac{f_L(t_{n+1})}{\mu} dt_n dt_{n+1} =$$

$$= \int\limits_{t_{n+1}=0}^{\infty} \frac{t_n^n}{(n)!\mu} f_L(t_{n+1}) dt_{n+1} = \frac{1}{n!\mu} \mathbb{E}(L^n)$$

□

There is one more equation about the $G_n(x)$'s that will be quite useful later:

Lemma 3.4. *if $n < \beta$ then*

$$\frac{d}{dt} G_n(t) = -G_{n-1}(t)$$

Proof.

$$\frac{d}{dt} G_n(t) = \frac{d}{dt} \int_{s=t}^{\infty} G_{n-1}(s) ds = \frac{d}{dt} \left[\frac{1}{(n-1)!} \mathbb{E}(T_0^{n-1}) - \int_{s=0}^{t} G_{n-1}(s) ds \right] =$$

$$= \frac{d}{dt} \left[- \int_{s=0}^{t} G_{n-1}(s) ds \right] = -G_{n-1}(t)$$

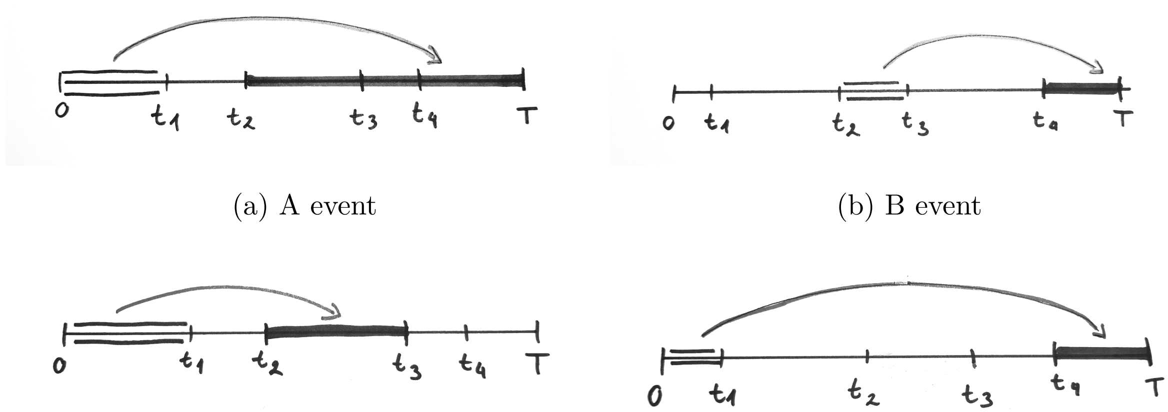
□

In this section our main goal is to calculate or approximate the 4th cumulant of the random variables S_T (defined in subsection 1). This requires the second and the fourth moment (as stated in Lemma 1.2). It is not easy to calculate the fourth moment of S_T with a general L tail distribution. However some general observations can be made about the integral that will bring us closer to computing $\mathcal{C}_4(S_T)$. To make this easier we introduce 4 events to simplify formulas.

Definition 3.5. *Let $0 < t_1 < t_2 < t_3 < t_4 < T$ be four numbers and let*

$$
\begin{aligned}
A &= \{H(t_1) > t_2 - t_1\} \\
B &= \{H(t_3) > t_4 - t_3\} \\
C &= \{t_3 - t_1 > H(t_1) > t_2 - t_1\} \\
D &= \{H(t_1) > t_4 - t_1\}
\end{aligned}
$$

be four events.



(a) A event

(b) B event

(c) C event

(d) D event

Figure 1: Events

It is important to note, that B and C are events concerning distinct renewal intervals.

Lemma 3.6. *([5], Appendix A) Without loss of generality let* $0 < t_1 < t_2 < T$*. Then*

$$
\mathbb{E}(S_T^2) = 2! \int\limits_{t_2=0}^{T} \int\limits_{t_1=0}^{t_2} \mathbb{P}(A) dt_1 dt_2
$$

Proof. We start with the expansion.

$$
\mathbb{E}(S_T^2) = 2! \int\limits_{t_2=0}^{T} \int\limits_{t_1=0}^{t_2} \mathbb{E}[\xi(t_1)\xi(t_2)] dt_1 dt_2
$$

Now $\mathbb{E}[\xi(t_1)\xi(t_2)]$ can be written in terms of conditional expectation, where we condition on the t_i's placement in relation with the arrival times. Because if t_1 and t_2 are in the same renewal period then the product of ξ_{t_1} and ξ_{t_2} will certainly be 1. (Indeed if $t_1, t_2 \in [T_{k-1}, T_k]$ then the respective ξ values are the same). However, if t_1 and t_2 are in different renewal blocks, then because of the independence of the ξ_i's, the product's expected value will become the expected values' product. Also from the distribution of ξ_i's the expected values are 0.

$$\begin{aligned}\mathbb{E}[\xi(t_1)\xi(t_2)] &= \mathbb{E}[\xi(t_1)\xi(t_2) \mid t_2 - t_1 > H(t_1)] \cdot \mathbb{P}(t_2 - t_1 > H(t_1)) \\ &+ \mathbb{E}[\xi(t_1)\xi(t_2) \mid H(t_1) > t_2 - t_1] \cdot \mathbb{P}(H(t_1) > t_2 - t_1) = \\ &= \mathbb{P}(H(t_1) > t_2 - t_1) = \mathbb{P}(A)\end{aligned}$$

Exactly because $t_2 - t_1 > H(t_1)$ means that ξ_{t_1} and ξ_{t_2} have values ξ_i with different i's. Meaning their product's expected value is 0. Also $\mathbb{E}[\xi(t_1)\xi(t_2) \mid H(t_1) > t_2 - t_1]$ is exactly 1 because from the condition follows: $\xi_{t_1} = \xi_{t_2} = \xi_{m(t_1)}$

□

Lemma 3.7. *Without loss of generality let* $0 < t_1 < t_2 < t_3 < t_4 < T$. *Then*

$$\mathbb{E}(S_T^4) = 4! \int_{t_4=0}^{T} \int_{t_3=0}^{t_4} \int_{t_2=0}^{t_3} \int_{t_1=0}^{t_2} \Big(\mathbb{P}(D) + \mathbb{P}(C \cap B)\Big) dt_1 dt_2 dt_3 dt_4$$

Proof.

$$\mathbb{E}(S_T^4) = 4! \int_{t_4=0}^{T} \int_{t_3=0}^{t_4} \int_{t_2=0}^{t_3} \int_{t_1=0}^{t_2} \mathbb{E}[\xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4)] dt_1 dt_2 dt_3 dt_4$$

This is very similar to the proof of lemma 3.6. With extra possible divisions of the t_i's. But the gist of it is the same. $\mathbb{E}[\xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4)]$ is only non-zero in those cases when there is an even number of t's in each renewal period, then their product will be positive. Because of independence if an odd number of t_i's are in one renewal period then the product's expected value will be zero (here then it is certainly true that there exists a t_i which sits alone in a renewal block, whose expected value is 0). It is easy to see that t_i's can only be placed in 2 ways for this product to have non-zero expected value. Either all four of them are in 1 big renewal period (D) or 2-2 of them are in two distinct renewal periods ($C \cap B$). From this follows:

$$\mathbb{E}[\xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4)] =$$
$$= \mathbb{E}[\xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4) \mid D\ ] \cdot \mathbb{P}(D) + \mathbb{E}[\xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4) \mid C \cap B\ ] \cdot \mathbb{P}(C \cap B)$$

Because of the conditions these expected values will be 1 , that leaves us with the sum of their probabilities, which is what we were looking for.

□

In the next subsections we will derive the exact values or estimates of $\mathcal{C}_4(S_T)$. Firstly in the exponential case, then turning to general tail distributions with $\beta > 4$ followed by the special case $\beta = 4$.

3.2 Exponential Case

Let L have tail distribution $\mathbb{P}(L > x) = e^{-\lambda x}$ ($\lambda > 0$). First , observe that the probabilities from lemma 3.6 and 3.7 are easily obtainable because of the properties of the Poisson point process.

$$\mathbb{P}(A) = \mathbb{P}(H(t_1) > t_2 - t_1) = e^{-\lambda(t_2 - t_1)} \tag{14}$$

$$\mathbb{P}(D) = \mathbb{P}(H(t_1) > t_4 - t_1) = e^{-\lambda(t_4 - t_1)}$$

Exactly because of the memoryless property of the exponential distribution:

$$\begin{aligned}\mathbb{P}(C \cap B) &= \mathbb{P}(\{H(t_3) > t_4 - t_3\} \cap \{t_3 - t_1 > H(t_1) > t_2 - t_1\}) = \\ &= \mathbb{P}(\{H(t_3) > t_4 - t_3\}\mathbb{P}(\{t_3 - t_1 > H(t_1) > t_2 - t_1\}) = \\ &= e^{-\lambda(t_4 - t_3)}\big(e^{-\lambda(t_2 - t_1)} - e^{-\lambda(t_3 - t_1)}\big) = \\ &= e^{-\lambda(t_4 - t_3 + t_2 - t_1)} - e^{-\lambda(t_4 - t_1)}\end{aligned}$$

Now from this:

$$\mathbb{P}(D) + \mathbb{P}(C \cap B) = e^{-\lambda(t_4 - t_3 + t_2 - t_1)} \tag{15}$$

Now with this in mind we can calculate $\mathcal{C}_4(S_T)$. First let's calculate $\mathbb{E}(S_T^4)$ with the help of lemma 3.7 and equation (15)

$$\frac{1}{4!}\mathbb{E}(S_T^4) = \int\limits_{t_4=0}^{T}\int\limits_{t_3=0}^{t_4}\int\limits_{t_2=0}^{t_3}\int\limits_{t_1=0}^{t_2} \Big(\mathbb{P}(D) + \mathbb{P}(C \cap B)\Big) dt_1 dt_2 dt_3 dt_4 =$$

$$= \int\limits_{t_4=0}^{T}\int\limits_{t_3=0}^{t_4}\int\limits_{t_2=0}^{t_3}\int\limits_{t_1=0}^{t_2} e^{-\lambda(t_4-t_3+t_2-t_1)} dt_1 dt_2 dt_3 dt_4 =$$

$$= \int\limits_{t_4=0}^{T} e^{-\lambda t_4} \int\limits_{t_3=0}^{t_4} e^{\lambda t_3} \int\limits_{t_2=0}^{t_3} e^{-\lambda t_2} \int\limits_{t_1=0}^{t_2} e^{\lambda t_1} dt_1 dt_2 dt_3 dt_4 =$$

$$= \frac{1}{\lambda}\int\limits_{t_4=0}^{T} e^{-\lambda t_4} \int\limits_{t_3=0}^{t_4} e^{\lambda t_3} \int\limits_{t_2=0}^{t_3} e^{-\lambda t_2} \left[e^{\lambda t_1}\right]_0^{t_2} dt_2 dt_3 dt_4 =$$

$$= \frac{1}{\lambda}\int\limits_{t_4=0}^{T} e^{-\lambda t_4} \int\limits_{t_3=0}^{t_4} e^{\lambda t_3} \int\limits_{t_2=0}^{t_3} e^{-\lambda t_2}(e^{\lambda t_2} - 1) dt_2 dt_3 dt_4 =$$

$$= \frac{1}{\lambda}\int\limits_{t_4=0}^{T} e^{-\lambda t_4} \int\limits_{t_3=0}^{t_4} e^{\lambda t_3} \int\limits_{t_2=0}^{t_3} (1 - e^{-\lambda t_2}) dt_2 dt_3 dt_4 =$$

$$= \frac{1}{\lambda}\int\limits_{t_4=0}^{T} e^{-\lambda t_4} \int\limits_{t_3=0}^{t_4} e^{\lambda t_3} \left[t_2 + \frac{1}{\lambda} e^{-\lambda t_2}\right]_0^{t_3} dt_3 dt_4 =$$

$$= \frac{1}{\lambda}\int\limits_{t_4=0}^{T} e^{-\lambda t_4} \int\limits_{t_3=0}^{t_4} (e^{\lambda t_3} t_3 + \frac{1}{\lambda} - \frac{1}{\lambda} e^{\lambda t_3}) dt_3 dt_4 =$$

$$= \frac{1}{\lambda}\int\limits_{t_4=0}^{T} e^{-\lambda t_4} \int\limits_{t_3=0}^{t_4} e^{\lambda t_3} t_3 dt_3 dt_4 + \frac{1}{\lambda^2}\int\limits_{t_4=0}^{T} e^{-\lambda t_4} \int\limits_{t_3=0}^{t_4} 1 dt_3 dt_4 + \frac{1}{\lambda^2}\int\limits_{t_4=0}^{T} e^{-\lambda t_4} \int\limits_{t_3=0}^{t_4} -e^{\lambda t_3} dt_3 dt_4$$

Lets name the three terms of this sum by I_1, I_2 and I_3 respectively. Lets solve them 1 by 1, first I_1 :

$$I_1 = \frac{1}{\lambda}\int\limits_{t_4=0}^{T} e^{-\lambda t_4} \int\limits_{t_3=0}^{t_4} e^{\lambda t_3} t_3 dt_3 dt_4 = \frac{1}{\lambda}\int\limits_{t_4=0}^{T} e^{-\lambda t_4} \left(\frac{1}{\lambda} e^{\lambda t_4} t_4 - \frac{1}{\lambda}\int\limits_{t_3=0}^{t_4} e^{\lambda t_3} dt_3\right) dt_4 =$$

$$= \frac{1}{\lambda^2}\int\limits_{t_4=0}^{T} \left(t_4 - e^{-\lambda t_4}\left[\frac{1}{\lambda} e^{\lambda t_3}\right]_0^{t_4}\right) dt_4 = \frac{1}{\lambda^2}\int\limits_{t_4=0}^{T} \Big(t_4 - \frac{1}{\lambda} + \frac{1}{\lambda} e^{-\lambda t_4}\Big) dt_4 =$$

$$= \frac{1}{\lambda^2}\left[\frac{1}{2}t_4^2 - \frac{1}{\lambda}t_4 - \frac{1}{\lambda^2}e^{-\lambda t_4}\right]_0^T = \frac{1}{2\lambda^2}T^2 - \frac{1}{\lambda^3}T - \frac{1}{\lambda^4}e^{-\lambda T} + \frac{1}{\lambda^4}$$

Now determine I_2:

$$I_2 = \frac{1}{\lambda^2}\int\limits_{t_4=0}^{T} e^{-\lambda t_4}\int\limits_{t_3=0}^{t_4} 1 dt_3 dt_4 = \frac{1}{\lambda^2}\int\limits_{t_4=0}^{T} e^{-\lambda t_4} t_4 dt_4 =$$

$$= \frac{1}{\lambda^2}\left(\left[\frac{-1}{\lambda}e^{-\lambda t_4}t_4\right]_0^T + \frac{1}{\lambda}\int\limits_{t_4=0}^{T} e^{-\lambda t_4}dt_4\right) = -\frac{1}{\lambda^3}e^{-\lambda T}T + \frac{1}{\lambda^3}\left[-\frac{1}{\lambda}e^{-\lambda t_4}\right]_0^T =$$

$$= -\frac{1}{\lambda^3}e^{-\lambda T}T - \frac{1}{\lambda^4}e^{-\lambda T} + \frac{1}{\lambda^4}$$

Lastly I_3:

$$I_3 = \frac{1}{\lambda^2}\int\limits_{t_4=0}^{T} e^{-\lambda t_4}\int\limits_{t_3=0}^{t_4} -e^{\lambda t_3}dt_3 dt_4 = \frac{1}{\lambda^2}\int\limits_{t_4=0}^{T} e^{-\lambda t_4}\left[\frac{-1}{\lambda}e^{\lambda t_3}\right]_0^{t_4} dt_4 =$$

$$= \frac{1}{\lambda^3}\int\limits_{t_4=0}^{T}\Big(-1 + e^{-\lambda t_4}\Big)dt_4 = \frac{1}{\lambda^3}\left[-t_4 - \frac{1}{\lambda}e^{-\lambda t_4}\right]_0^T = -\frac{1}{\lambda^3}T - \frac{1}{\lambda^4}e^{-\lambda T} + \frac{1}{\lambda^4}$$

Now putting all this together we have that :

$$\begin{aligned}
&\frac{1}{4!}\mathbb{E}(S_T^4) = I_1 + I_2 + I_3 \\
&= \frac{1}{2\lambda^2}T^2 - \frac{1}{\lambda^3}T - \frac{1}{\lambda^4}e^{-\lambda T} + \frac{1}{\lambda^4} - \frac{1}{\lambda^3}e^{-\lambda T}T - \frac{1}{\lambda^4}e^{-\lambda T} + \frac{1}{\lambda^4} - \frac{1}{\lambda^3}T - \frac{1}{\lambda^4}e^{-\lambda T} + \frac{1}{\lambda^4} \\
&= \frac{1}{2\lambda^2}T^2 - \frac{2}{\lambda^3}T + \frac{3}{\lambda^4} - \frac{1}{\lambda^3}e^{-\lambda T}T - \frac{3}{\lambda^4}e^{-\lambda T} = \frac{1}{2\lambda^2}T^2 - \frac{2}{\lambda^3}T + \frac{3}{\lambda^4} + o(1)
\end{aligned}$$

Since T is a big number, that is what we are assuming, then $e^{-\lambda T}$ goes to zero exponentially. From this we can conclude that.

$$\mathbb{E}(S_T^4) = \frac{12}{\lambda^2}T^2 - \frac{48}{\lambda^3}T + \frac{72}{\lambda^4} + o(1) \tag{16}$$

Now lets focus on $\mathbb{E}(S_T^2)$. Using lemma 3.6 and equation (14) we can write:

$$\frac{1}{2!}\mathbb{E}(S_T^2) = \int\limits_{t_2=0}^{T}\int\limits_{t_1=0}^{t_2} \mathbb{P}(A)dt_1 dt_2 = \int\limits_{t_2=0}^{T}\int\limits_{t_1=0}^{t_2} e^{-\lambda(t_2-t_1)}dt_1 dt_2 = \int\limits_{t_2=0}^{T} e^{-\lambda t_2}\int\limits_{t_1=0}^{t_2} e^{\lambda t_1}dt_1 dt_2$$

Notice that this is nothing else then $-\lambda^2 I_3$. Meaning:

$$\mathbb{E}(S_T^2) = -(2!)\lambda^2 \left(-\frac{1}{\lambda^3}T - \frac{1}{\lambda^4}e^{-\lambda T} + \frac{1}{\lambda^4}\right) = \frac{2}{\lambda}T + \frac{2}{\lambda^2}e^{-\lambda T} - \frac{2}{\lambda^2}$$

Similarly to the case of $\mathbb{E}(S_T^4)$, all the elements that have $e^{-\lambda T}$ in them go to zero exponentially. Applying this we get:

$$\mathbb{E}(S_T^2) = \frac{2}{\lambda}T - \frac{2}{\lambda^2} + o(1) \tag{17}$$

Now we have all the ingredients to express $\mathcal{C}_4(S_T)$. From lemma 1.2 and equations (17) and (16) we get that:

$$\begin{aligned}\mathcal{C}_4(S_T) = \mathbb{E}(X^4) - 3(\mathbb{E}(X^2))^2 &= \left(\frac{12}{\lambda^2}T^2 - \frac{48}{\lambda^3}T + \frac{72}{\lambda^4}\right) - 3\left(\frac{2}{\lambda}T - \frac{2}{\lambda^2}\right)^2 + o(1) \\ &= \frac{12}{\lambda^2}T^2 - \frac{48}{\lambda^3}T + \frac{72}{\lambda^4} - \frac{12}{\lambda^2}T^2 + \frac{24}{\lambda^3}T - \frac{12}{\lambda^4} + o(1)\end{aligned}$$

By simplifying we get that the T^2 terms cancel out, leaving us with an expression whose leading order is $\sim T$:

$$\mathcal{C}_4(S_T) = -\frac{24}{\lambda^3}T + \frac{60}{\lambda^4} + o(1) \tag{18}$$

For future reference we also include an overall simple relation:

Lemma 3.8. *In the case where L has distribution Exp(λ) ($\lambda > 0$):*

$$G_n(0) = \frac{1}{\lambda^{n-1}}$$

Proof. First by the help of lemma 3.3 it is clear that:

$$G_n(0) = \frac{1}{n!\mu}\mathbb{E}(L^n)$$

Since L has exponential distribution then $\mathbb{E}(L) = \mu = \frac{1}{\lambda}$. Now we only need $\mathbb{E}(L^n)$:

$$\mathbb{E}(L^n) = \int_0^\infty x^n e^{-\lambda x} dx = \left[-\frac{1}{\lambda}x^n e^{-\lambda x}\right]_0^T + \int_0^\infty \frac{n}{\lambda}x^{n-1}e^{-\lambda x}dx = \frac{n}{\lambda}\mathbb{E}(L^{n-1})$$

Now by induction it is clear that:

$$\mathbb{E}(L^n) = \frac{n!}{\lambda^n}\mathbb{E}(L^0) = \frac{n!}{\lambda^n}$$

Writing it back into the equation with $G_n(0)$ we get:

$$G_n(0) = \frac{1}{n!\mu}\mathbb{E}(L^n) = \frac{\lambda}{n!}\frac{n!}{\lambda^n} = \frac{1}{\lambda^{n-1}}$$

□

3.3 Polynomial Case ($\beta > 4$)

These calculations were exploratory, for a Poisson point process we could calculate everything exactly. When we consider more general tail distributions the events A, B, C, D will have to be handled more carefully as the nice properties of the exponential distribution do not apply in general.

We will use the approximation that B and C are regarded independent. Although an approximation, this is still a reasonable assumption, justified by the following argument. If $T \to \infty$ the distance between t_2 and t_3 is growing larger and larger. When this distance is huge then the two events have almost negligible effect on each other. During the time $t_3 - t_2$ the renewal process "almost completely forgets the past", that is $H(t_3)$ can be assumed to be independent of $H(t_2)$ and $H(t_1)$, meaning that B is independent of C. Remember that B and C are events concerning distinct renewal intervals. (See figure 1 on page 15)

Now let's consider the more general case, when L has a polynomial tail distribution $\mathbb{P}(L > x) \sim ax^{-\beta}$ as $x \to \infty$. In the next 2 subsections we will be heavily relying on the notations of $G_n(x)$ (Definition 3.1).

However we must handle the $G_n(x)$'s carefully as not all of them exist. By lemma 3.2 we know that

$$G_n(x) = c_1 x^{-(\beta-n)} + \mathcal{O}(x^{-(\beta-n+1)}) \qquad (n < \beta)$$

The constant's exact value here is irrelevant. From this we know that for $n < \beta$, $G_n(x)$ certainly exists since we integrate G_{n-1} to get it and here:

$$G_{n-1}(x) = c_2 x^{-(\beta-n+1)} + \mathcal{O}(x^{-(\beta-n+2)})$$

Where $x^{-(\beta-n+1)}$ is integrable , since $\beta - n + 1 > 1$. So if $\beta > 4$ then $G_n(x)$'s exist for $n \leq 4 < \beta$. So we can use them without any problem.

Let's start by calculating $\mathbb{E}(S_T^2)$, where according to lemma 3.6:

$$\frac{1}{2!}\mathbb{E}(S_T^2) = \int\limits_{t_2=0}^{T}\int\limits_{t_1=0}^{t_2} \mathbb{E}[\xi(t_1)\xi(t_2)]dt_1dt_2 = \int\limits_{t_2=0}^{T}\int\limits_{t_1=0}^{t_2} \mathbb{P}(A)dt_1dt_2 =$$

because of stationarity (3) and the definition of $G_n(k)$ (def 3.1)

$$= \int\limits_{t_2=0}^{T}\int\limits_{t_1=0}^{t_2} \mathbb{P}(T_0 > t_2 - t_1)dt_1dt_2 = \int\limits_{t_2=0}^{T}\int\limits_{t_1=0}^{t_2} G_1(t_2 - t_1)dt_1dt_2 =$$

With the substitution $s = t_2 - t_1$ and by the definition of $G_2(k)$ (def 3.1)

$$= \int\limits_{t_2=0}^{T}\int\limits_{s=0}^{t_2} G_1(s)dsdt_2 = \int\limits_{t_2=0}^{T} \Big(G_2(0) - G_2(t_2)\Big)dt_2 = TG_2(0) - (G_3(0) - G_3(T))$$

So we can conclude that :

$$\mathbb{E}(S_T^2) = 2\left(TG_2(0) - G_3(0) + G_3(T)\right) \tag{19}$$

Using lemma 3.8 we can compare this to the already calculated exponential $\mathbb{E}(S_T^2)$ in (17). In this case:

$$\mathbb{E}(S_T^2) = 2\left(TG_2(0) - G_3(0) + G_3(T)\right) = \frac{2}{\lambda}T - \frac{2}{\lambda^2} + o(1)$$

In accordance with (17). (We used here that in the exponential case $G_3(T) \sim e^{-constant*T}$ $= o(1)$).

Here it is worth noting that during the argument for (19), we did not use that $\beta > 4$. In fact this equation stays true for $\beta = 4$ and we will refer back to it in Subsection 3.4. Also I would like to mention that in [5] similar calculations are done however, without the $G_n(x)$ notation. So our argument for $\mathbb{E}(S_T^2)$ is depending on the existence of $G_3(x)$ and $G_2(x)$, to be more precise:

- for $\beta > 3$ (19) is true

- for $3 \geq \beta > 2$ this argument stands, save $G_3(x)$'s existence. There we should use:

$$\int_{t_2=0}^{T} \Big(G_2(0) - G_2(t_2)\Big) dt_2 = TG_2(0) + o(T^{3-\beta}) = TG_2(0) + o(T)$$

- for $\beta = 2$ the explicit argument in [5] gives us:

$$\mathbb{E}(S_T^2) = \frac{2a}{\mu} T \ln(T) + \mathcal{O}(T)$$

Which is exactly the doubling effect mentioned in section 1.4.

This concludes our discussion of $\mathbb{E}(S_T^2)$.

Now lets focus on the rather complicated $\mathbb{E}(S_T^4)$. The calculations are elementary however there is a lot of them and as we approach the end result , we will be dividing it into smaller integrals, solving it one bit at a time. But first let's draw the big picture using lemma 3.7:

$$\frac{1}{4!}\mathbb{E}(S_T^4) = \int_{t_4=0}^{T}\int_{t_3=0}^{t_4}\int_{t_2=0}^{t_3}\int_{t_1=0}^{t_2} \mathbb{E}[\xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4)] dt_1 dt_2 dt_3 dt_4 =$$

$$= \int_{t_4=0}^{T}\int_{t_3=0}^{t_4}\int_{t_2=0}^{t_3}\int_{t_1=0}^{t_2} \Bigg(\mathbb{P}(D) + \mathbb{P}(C \cap B)\Bigg) dt_1 dt_2 dt_3 dt_4 =$$

$$= \int_{t_4=0}^{T}\int_{t_3=0}^{t_4}\int_{t_2=0}^{t_3}\int_{t_1=0}^{t_2} \mathbb{P}(D) dt_1 dt_2 dt_3 dt_4 + \int_{t_4=0}^{T}\int_{t_3=0}^{t_4}\int_{t_2=0}^{t_3}\int_{t_1=0}^{t_2} \mathbb{P}(C \cap B) dt_1 dt_2 dt_3 dt_4 \qquad (20)$$

The case of $\mathbb{P}(D)$:

Let's start by solving the term involving $\mathbb{P}(D)$.

$$\int_{t_4=0}^{T}\int_{t_3=0}^{t_4}\int_{t_2=0}^{t_3}\int_{t_1=0}^{t_2} \mathbb{P}(D) dt_1 dt_2 dt_3 dt_4 = \int_{t_4=0}^{T}\int_{t_3=0}^{t_4}\int_{t_2=0}^{t_3}\int_{t_1=0}^{t_2} \mathbb{P}(H(t_1) > t_4 - t_1) dt_1 dt_2 dt_3 dt_4 =$$

because of the stationarity of T_0 (3)

$$= \int_{t_4=0}^{T}\int_{t_3=0}^{t_4}\int_{t_2=0}^{t_3}\int_{t_1=0}^{t_2} \mathbb{P}(T_0 > t_4 - t_1) dt_1 dt_2 dt_3 dt_4 =$$

by the definiton of $G_n(k)$'s (def 3.1)

$$= \int\limits_{t_4=0}^{T} \int\limits_{t_3=0}^{t_4} \int\limits_{t_2=0}^{t_3} \int\limits_{t_1=0}^{t_2} G_1(t_4 - t_1) dt_1 dt_2 dt_3 dt_4 =$$

by $s = t_4 - t_1$ substitution:

$$= \int\limits_{t_4=0}^{T} \int\limits_{t_3=0}^{t_4} \int\limits_{t_2=0}^{t_3} \int\limits_{s=t_4-t_2}^{t_4} G_1(s) ds dt_2 dt_3 dt_4 = \int\limits_{t_4=0}^{T} \int\limits_{t_3=0}^{t_4} \int\limits_{t_2=0}^{t_3} \Big(G_2(t_4 - t_2) - G_2(t_4) \Big) dt_2 dt_3 dt_4 =$$

by $u = t_4 - t_2$ substitution:

$$= \int\limits_{t_4=0}^{T} \int\limits_{t_3=0}^{t_4} \left(-G_2(t_4) t_3 + \int\limits_{u=t_4-t_3}^{t_4} G_2(u) du \right) dt_3 dt_4 =$$

$$= \int\limits_{t_4=0}^{T} \int\limits_{t_3=0}^{t_4} [-G_2(t_4) t_3 + G_3(t_4 - t_3) - G_3(t_4)] dt_3 dt_4 =$$

by $v = t_4 - t_3$ substitution:

$$= \int\limits_{t_4=0}^{T} \left(-\frac{1}{2} G_2(t_4) t_4^2 - G_3(t_4) t_4 + \int\limits_{v=0}^{t_4} G_3(v) dv \right) dt_4 = \tag{21}$$

$$= - \int\limits_{t_4=0}^{T} \frac{1}{2} G_2(t_4) t_4^2 dt_4 - \int\limits_{t_4=0}^{T} G_3(t_4) t_4 dt_4 + \int\limits_{t_4=0}^{T} [G_4(0) - G_4(t_4)] dt_4 =$$

$$= T G_4(0) - \int\limits_{t_4=0}^{T} \frac{1}{2} G_2(t_4) t_4^2 dt_4 - \int\limits_{t_4=0}^{T} G_3(t_4) t_4 dt_4 - \int\limits_{t_4=0}^{T} G_4(t_4) dt_4 =$$

$$= T G_4(0) - U_1 - U_2 - U_3$$

Where the names are:

$$U_1 = \int\limits_{t_4=0}^{T} \frac{1}{2} G_2(t_4) t_4^2 dt_4$$

$$U_2 = \int\limits_{t_4=0}^{T} G_3(t_4) t_4 dt_4$$

$$U_3 = \int\limits_{t_4=0}^{T} G_4(t_4)dt_4$$

Solving them 1 by 1 with integration by parts and using lemma 3.4:

$$U_1 = \int\limits_{t_4=0}^{T} \frac{1}{2}G_2(t_4)t_4^2dt_4 = \left[-\frac{1}{2}G_3(t_4)t_4^2\right]_0^T + \int\limits_{t_4=0}^{T} G_3(t_4)t_4dt_4 = -\frac{1}{2}T^2G_3(T) + U_2$$

$$U_2 = \int\limits_{t_4=0}^{T} G_3(t_4)t_4dt_4 = [-G_4(t_4)t_4]_0^T + \int\limits_{t_4=0}^{T} G_4(t_4)dt_4 = -TG_4(T) + U_3$$

$$U_3 = \int\limits_{t_4=0}^{T} G_4(t_4)dt_4 = G_5(0) - G_5(T)$$

At this point we discuss the $\beta = 5$, $\beta > 5$ and $4 < \beta < 5$ cases separately. Technically here we assume that $G_5(x)$ exists, which would require $\beta > 5$ but the last step is replaceable for $4 < \beta \leq 5$. The trick with $G_5(0) - G_5(T)$ doesn't work for $4 < \beta \leq 5$ because $G_5(y) = \int\limits_{x=y}^{\infty} G_4(x)dx$ and here by lemma 3.2 $G_4(x) \sim x^{-(\beta-4)} \sim x^{-t}$ where $t \in (0, 1]$ $(t = \beta - 4)$ is not integrable. Instead of writing it as a combination of $G_5(y)$'s, an approximation can be given with the help of lemma 3.2.

$$\int\limits_{t=0}^{T} G_4(x)dx = \mathcal{O}(1) + \int\limits_{x=1}^{T} \left[\frac{a}{\mu \prod_{i=1}^{n}(\beta - i)}x^{-(\beta-n)} + \mathcal{O}(x^{-(\beta-n+1)})\right]dx =$$

- For $\beta > 5$ the simple iteration works as $G_5(x)$ exist, so already without the approximation lemma it is true that:

$$= G_5(0) - G_5(T)$$

- With $t \in (0, 1)$ when $4 < \beta < 5$ it is true that:

$$= \int\limits_{x=1}^{T} \left[c_1x^{-t} + \mathcal{O}(x^{-(t+1)})\right]dx + \mathcal{O}(1) = c_1T^{1-t} + \mathcal{O}(1)$$

- With $t = 1$, when $\beta = 5$ being a special case:

$$= \int\limits_{x=1}^{T} \left[c_1x^{-t} + \mathcal{O}(x^{-(t+1)})\right]dx + \mathcal{O}(1) = c_2 \ln(T) + \mathcal{O}(1)$$

as $T \to \infty$. So U_3 will be one of the expressions above, depending on β. We note that in all cases $U_3 = o(T)$

Thus, we obtain for general $\beta > 4$ (by using the U_3 approximation):

$$\int_{t_4=0}^{T}\int_{t_3=0}^{t_4}\int_{t_2=0}^{t_3}\int_{t_1=0}^{t_2} \mathbb{P}(D)dt_1dt_2dt_3dt_4 = TG_4(0) - \frac{1}{2}T^2G_3(T) - 2TG_4(T) + o(T) \tag{22}$$

This holds for all $\beta > 4$

<u>The case of $\mathbb{P}(C \cap B)$:</u>

Now we are going to solve the integral containing $\mathbb{P}(C \cap B)$. Here we will assume that C and B are independent , however it is not necessarily true in general, this is a strong assumption, but it makes the calculations doable. And for our demonstrating purposes it will suffice.

But in fact if we know that an arrival happened just now or long ago, then prior knowledge will ,in most cases ,change the next arrival's distribution. This is easier to see for distributions whose domain is bounded. (e.g: Uniform on [0,1] if I know that the last arrival happened at 0.9 ago, then it is certain that in the next 0.1 it will happen again. On the other hand if it happened 0.5 ago, then on the next 0.1 it will only happen with 20% probability)

Nevertheless, these assumptions still allow us to observe the behavior of these processes , albeit not giving us exact results for the moments and cumulants of S_T's in case of a general polynomial tail distribution.

The case of $\mathbb{P}(C \cap B)$ assuming the independence of B and C:

$$\int_{t_4=0}^{T}\int_{t_3=0}^{t_4}\int_{t_2=0}^{t_3}\int_{t_1=0}^{t_2} \mathbb{P}(C \cap B)dt_1dt_2dt_3dt_4 = \int_{t_4=0}^{T}\int_{t_3=0}^{t_4}\int_{t_2=0}^{t_3}\int_{t_1=0}^{t_2} \mathbb{P}(C)\mathbb{P}(B)dt_1dt_2dt_3dt_4 =$$

$$= \int_{t_4=0}^{T}\int_{t_3=0}^{t_4}\int_{t_2=0}^{t_3}\int_{t_1=0}^{t_2} \mathbb{P}(H(t_3) > t_4 - t_3)\mathbb{P}(t_3 - t_1 > H(t_1) > t_2 - t_1)dt_1dt_2dt_3dt_4 =$$

because of stationarity of T_0 (3)

$$= \int_{t_4=0}^{T}\int_{t_3=0}^{t_4}\int_{t_2=0}^{t_3}\int_{t_1=0}^{t_2} \mathbb{P}(T_0 > t_4 - t_3)\mathbb{P}(t_3 - t_1 > T_0 > t_2 - t_1)dt_1dt_2dt_3dt_4 =$$

Also using the definition of $G_n(k)$'s (def 3.1):

$$= \int\limits_{t_4=0}^{T} \int\limits_{t_3=0}^{t_4} \int\limits_{t_2=0}^{t_3} \int\limits_{t_1=0}^{t_2} G_1(t_4-t_3)\left[G_1(t_2-t_1)-G_1(t_3-t_1)\right] dt_1 dt_2 dt_3 dt_4 =$$

$$= \int\limits_{t_4=0}^{T} \int\limits_{t_3=0}^{t_4} \int\limits_{t_2=0}^{t_3} \int\limits_{t_1=0}^{t_2} \Big[G_1(t_4-t_3)G_1(t_2-t_1) - G_1(t_4-t_3)G_1(t_3-t_1)\Big] dt_1 dt_2 dt_3 dt_4$$

Before we start calculating, an interesting observation is that in the exponential case $G_1(t_4-t_3)G_1(t_3-t_1)$ simplifies significantly and in fact is equal to $\mathbb{P}(D)$. They cancel each other out, leaving us with one 4 dimensional integral, that we solved.

Here let's start introducing notations for the subsequent terms:

$$I = \int\limits_{t_4=0}^{T} \int\limits_{t_3=0}^{t_4} \int\limits_{t_2=0}^{t_3} \int\limits_{t_1=0}^{t_2} G_1(t_4-t_3)G_1(t_2-t_1) dt_1 dt_2 dt_3 dt_4$$

$$II = \int\limits_{t_4=0}^{T} \int\limits_{t_3=0}^{t_4} \int\limits_{t_2=0}^{t_3} \int\limits_{t_1=0}^{t_2} G_1(t_4-t_3)G_1(t_3-t_1) dt_1 dt_2 dt_3 dt_4$$

Rewriting our result we get that :

$$\int\limits_{t_4=0}^{T} \int\limits_{t_3=0}^{t_4} \int\limits_{t_2=0}^{t_3} \int\limits_{t_1=0}^{t_2} \mathbb{P}(C \cap B) dt_1 dt_2 dt_3 dt_4 = I - II \tag{23}$$

We will start by calculating I:

$$I = \int\limits_{t_4=0}^{T} \int\limits_{t_3=0}^{t_4} \int\limits_{t_2=0}^{t_3} \int\limits_{t_1=0}^{t_2} G_1(t_4-t_3)G_1(t_2-t_1) dt_1 dt_2 dt_3 dt_4 =$$

$$= \int\limits_{t_4=0}^{T} \int\limits_{t_3=0}^{t_4} G_1(t_4-t_3) \int\limits_{t_2=0}^{t_3} \int\limits_{t_1=0}^{t_2} G_1(t_2-t_1) dt_1 dt_2 dt_3 dt_4 =$$

observe that $\int\limits_{t_2=0}^{t_3} \int\limits_{t_1=0}^{t_2} G_1(t_2-t_1) dt_1 dt_2$ is nothing else, than $\frac{1}{2!}\mathbb{E}(S_{t_3}^2)$. So by (19)

$$I = \int\limits_{t_4=0}^{T} \int\limits_{t_3=0}^{t_4} G_1(t_4-t_3)\left(t_3 G_2(0) - G_3(0) + G_3(t_3)\right) dt_3 dt_4 = I_1 - I_2 + I_3$$

Now here we have 3 integrals, where I_1, I_2 and I_3 stands for :

$$I_1 = \int_{t_4=0}^{T} \int_{t_3=0}^{t_4} G_1(t_4 - t_3) t_3 G_2(0) dt_3 dt_4$$

$$I_2 = \int_{t_4=0}^{T} \int_{t_3=0}^{t_4} G_1(t_4 - t_3) G_3(0) dt_3 dt_4$$

$$I_3 = \int_{t_4=0}^{T} \int_{t_3=0}^{t_4} G_1(t_4 - t_3) G_3(t_3) dt_3 dt_4$$

Then also decompose II:

$$II = \int_{t_4=0}^{T} \int_{t_3=0}^{t_4} \int_{t_2=0}^{t_3} \int_{t_1=0}^{t_2} G_1(t_4 - t_3) G_1(t_3 - t_1) dt_1 dt_2 dt_3 dt_4 =$$

$$= \int_{t_4=0}^{T} \int_{t_3=0}^{t_4} G_1(t_4 - t_3) \int_{t_2=0}^{t_3} \int_{t_1=0}^{t_2} G_1(t_3 - t_1) dt_1 dt_2 dt_3 dt_4$$

We need to calculate the inner integral first (substitute $s = t_3 - t_1$):

$$\int_{t_2=0}^{t_3} \int_{t_1=0}^{t_2} G_1(t_3 - t_1) dt_1 dt_2 = \int_{t_2=0}^{t_3} \int_{s=t_3-t_2}^{t_3} G_1(s) ds dt_2 = \int_{t_2=0}^{t_3} \Big[G_2(t_3 - t_2) - G_2(t_3) \Big] dt_2 =$$

substituting $u = t_3 - t_2$:

$$= \int_{u=0}^{t_3} G_2(u) du - G_2(t_3) t_3 = G_3(0) - G_3(t_3) - t_3 G_2(t_3)$$

So now we know that:

$$II = \int_{t_4=0}^{T} \int_{t_3=0}^{t_4} G_1(t_4 - t_3) \left[G_3(0) - G_3(t_3) - t_3 G_2(t_3) \right] dt_3 dt_4 =$$

$$\int_{t_4=0}^{T} \int_{t_3=0}^{t_4} G_1(t_4 - t_3) G_3(0) dt_3 dt_4 - \int_{t_4=0}^{T} \int_{t_3=0}^{t_4} G_1(t_4 - t_3) G_3(t_3) dt_3 dt_4$$

$$-\int_{t_4=0}^{T}\int_{t_3=0}^{t_4} G_1(t_4-t_3)t_3G_2(t_3)dt_3dt_4 = II_1 - II_2 - II_3$$

Where the names are:

$$II_1 = \int_{t_4=0}^{T}\int_{t_3=0}^{t_4} G_1(t_4-t_3)G_3(0)dt_3dt_4$$

$$II_2 = \int_{t_4=0}^{T}\int_{t_3=0}^{t_4} G_1(t_4-t_3)G_3(t_3)dt_3dt_4$$

$$II_3 = \int_{t_4=0}^{T}\int_{t_3=0}^{t_4} G_1(t_4-t_3)t_3G_2(t_3)dt_3dt_4$$

Now first let's make the observations that:

$$I_2 = II_1 \ and \ I_3 = II_2$$

From this and (23) follows, that :

$$\int_{t_4=0}^{T}\int_{t_3=0}^{t_4}\int_{t_2=0}^{t_3}\int_{t_1=0}^{t_2} \mathbb{P}(C \cap B)dt_1dt_2dt_3dt_4 = (I_1 - I_2 + I_3) - (II_1 - II_2 - II_3) =$$

$$= I_1 - 2I_2 + 2I_3 + II_3 \tag{24}$$

We need to compute these four integrals, starting with I_1:

$$I_1 = \int_{t_4=0}^{T}\int_{t_3=0}^{t_4} G_1(t_4-t_3)t_3G_2(0)dt_3dt_4 = G_2(0)\int_{t_4=0}^{T}\int_{t_3=0}^{t_4} G_1(t_4-t_3)t_3dt_3dt_4 =$$

Substitute $s = t_4 - t_3$:

$$G_2(0)\int_{t_4=0}^{T}\int_{s=0}^{t_4} G_1(s)(t_4-s)dsdt_4 =$$

$$= G_2(0)\left(\int_{t_4=0}^{T} t_4\int_{s=0}^{t_4} G_1(s)dsdt_4 - \int_{t_4=0}^{T}\int_{s=0}^{t_4} sG_1(s)dsdt_4\right) =$$

$$= G_2(0)\int\limits_{t_4=0}^{T}\Big[t_4G_2(0) - t_4G_2(t_4)\Big]dt_4 - G_2(0)\int\limits_{t_4=0}^{T}\int\limits_{s=0}^{t_4} sG_1(s)dsdt_4 =$$

Integration by parts and using lemma 3.4

$$= G_2^2(0)\int\limits_{t_4=0}^{T} t_4dt_4 - G_2(0)\int\limits_{t_4=0}^{T} t_4G_2(t_4)dt_4$$

$$-G_2(0)\int\limits_{t_4=0}^{T}\left(\Big[-sG_2(s)\Big]_0^{t_4} + \int\limits_{s=0}^{t_4} G_2(s)ds\right)dt_4 =$$

$$= G_2^2(0)\frac{1}{2}T^2 - G_2(0)\int\limits_{t_4=0}^{T} t_4G_2(t_4)dt_4$$

$$+G_2(0)\int\limits_{t_4=0}^{T} t_4G_2(t_4)dt_4 - G_2(0)\int\limits_{t_4=0}^{T}[G_3(0) - G_3(t_4)]dt_4 = \tag{25}$$

Canceling out the two integrals

$$= G_2^2(0)\frac{1}{2}T^2 - G_2(0)TG_3(0) + G_2(0)G_4(0) - G_2(0)G_4(T) \tag{26}$$

With that finished , let's solve I_2 (which we already have).By noticing that :

$$I_2 = \int\limits_{t_4=0}^{T}\int\limits_{t_3=0}^{t_4} G_1(t_4 - t_3)G_3(0)dt_3dt_4 = G_3(0)\frac{1}{2!}\mathbb{E}(S_T^2)$$

$$= G_3(0)[TG_2(0) - G_3(0) + G_3(T)] \tag{27}$$

follows from (19)

Now only I_3 and II_3 are left to be calculated. However we are going to approximate them since as it turns out their leading terms are negligible to us. For this we need to notice 2 facts. First, the 2 integrals are very similar in fact $G_3(t)$ and $tG_2(t)$ are both of the same order. So there exist a constant C for which $tG_2(t) < CG_3(t)$ $(\forall t > 0)$ So by this we can also conclude that:

$$II_2 = \int\limits_{t_4=0}^{T}\int\limits_{t_3=0}^{t_4} G_1(t_4 - t_3)t_3G_2(t_3)dt_3dt_4 < \int\limits_{t_4=0}^{T}\int\limits_{t_3=0}^{t_4} G_1(t_4 - t_3)CG_3(t_3)dt_3dt_4 =$$

$$= C \int_{t_4=0}^{T} \int_{t_3=0}^{t_4} G_1(t_4 - t_3) G_3(t_3) dt_3 dt_4 = C \cdot I_3$$

Now we have that $2I_3 + II_3 < (2 + C) \cdot I_3$.

By rewriting the original I_3 integral with integration by parts we see that:

$$I_3 = \int_{t_4=0}^{T} \int_{t_3=0}^{t_4} G_1(t_4 - t_3) G_3(t_3) dt_3 dt_4 =$$

$$= \int_{t_4=0}^{T} \left(\Big[G_2(t_4 - t_3) G_3(t_3) \Big]_{t_3=0}^{t_4} + \int_{t_3=0}^{t_4} G_2(t_4 - t_3) G_2(t_3) dt_3 \right) dt_4 =$$

$$= \int_{t_4=0}^{T} \Big[G_2(0) G_3(t_4) - G_2(t_4) G_3(0) \Big] dt_4 + \int_{t_4=0}^{T} \int_{t_3=0}^{t_4} G_2(t_4 - t_3) G_2(t_3) dt_3 dt_4 = \tag{28}$$

$$= G_2(0) \Big(G_4(0) - G_4(T) \Big) - G_3(0) \Big(G_3(0) - G_3(T) \Big) + \int_{t_4=0}^{T} \int_{t_3=0}^{t_4} G_2(t_4 - t_3) G_2(t_3) dt_3 dt_4 =$$

$$= \int_{t_4=0}^{T} \int_{t_3=0}^{t_4} G_2(t_4 - t_3) G_2(t_3) dt_3 dt_4 + \mathcal{O}(1)$$

This last step is true as $G_3(T) = \mathcal{O}(1)$, $G_4(T) = \mathcal{O}(1)$ (from lemma 3.2).

The second observation is that the inner part of this last double integral is a convolution of $G_2(x)$ with itself. Convolution has the property that it is a closed operation in L^1 space. That is if $f, g \in L^1$ are convolved, then the resulting function will also be in L^1. And since $G_3(0)$ exists (for $\beta > 3$) it is true that $G_2(x)$ is integrable , thus the convolution is also in L^1, let's call it $Z(t_4)$. And thus:

$$\int_{t_4=0}^{T} \int_{t_3=0}^{t_4} G_2(t_4 - t_3) G_2(t_3) dt_3 dt_4 = \int_{t_4=0}^{T} Z(t_4) dt_4 = \mathcal{O}(1)$$

With this we can see that:

$$I_3 = \int_{t_4=0}^{T} \int_{t_3=0}^{t_4} G_2(t_4 - t_3) G_2(t_3) dt_3 dt_4 + \mathcal{O}(1) = \mathcal{O}(1) + \mathcal{O}(1) = o(T)$$

From this it is clear that:

$$2I_3 + II_3 < (2 + C) \cdot I_3 = (2 + C) \cdot o(T) = o(T) \tag{29}$$

So this means that finally we have everything to finish calculating the integral of $P(C \cap B)$. By using (26), (27) and (29):

$$\int_{t_4=0}^{T}\int_{t_3=0}^{t_4}\int_{t_2=0}^{t_3}\int_{t_1=0}^{t_2} \mathbb{P}(C \cap B) dt_1 dt_2 dt_3 dt_4 = I_1 + (-2)I_2 + 2I_3 + II_3 =$$

$$G_2^2(0)\frac{1}{2}T^2 - G_2(0)TG_3(0) + G_2(0)G_4(0) - G_2(0)G_4(T) +$$

$$+(-2)G_3(0)[TG_2(0) - G_3(0) + G_3(T)] + o(T) = \tag{30}$$

$$G_2^2(0)\frac{1}{2}T^2 - 3G_2(0)G_3(0)T + o(T)$$

as $G_3(T) = o(T^{-1})$ and $G_4(T) = o(1)$ so are all of $o(T)$,when $\beta > 4$ (by lemma 3.2)

The fourth cumulant $\mathcal{C}_4(S_T)$:

Now we have all that we need to express $\mathcal{C}_4(S_T)$. Lets calculate the 4th moment with (20), (22) and (30) :

$$\mathbb{E}(S_T^4) = 24\Bigg(TG_4(0) - \frac{1}{2}T^2G_3(T) - 2TG_4(T) + o(T) +$$

$$+G_2^2(0)\frac{1}{2}T^2 - G_2(0)TG_3(0) + G_2(0)G_4(0) - G_2(0)G_4(T) +$$

$$+(-2)G_3(0)[TG_2(0) - G_3(0) + G_3(T)] + o(T)\Bigg) =$$

Here it is worth noting that since $\beta > 4$ by lemma 3.2. $G_2(T) = o(T^{-2})$ $G_3(T) = o(T^{-1})$ and $G_4(T) = o(1)$. In particular they are all $o(T)$. So:

$$= 24\Big(TG_4(0) - \frac{1}{2}T^2G_3(T) + G_2^2(0)\frac{1}{2}T^2 - G_2(0)TG_3(0) - 2TG_3(0)G_2(0) + o(T)\Big) =$$

$$= 24\Big(TG_4(0) - \frac{1}{2}T^2G_3(T) + G_2^2(0)\frac{1}{2}T^2 - 3G_2(0)TG_3(0) + o(T)\Big)$$

Now onto the second moment, from (19) we can write:

$$3(\mathbb{E}(S_T^2))^2 = 3\Big[2\left(TG_2(0) - G_3(0) + G_3(T)\right)\Big]^2 =$$

$$= 12\Big(T^2G_2^2(0) + G_3^2(0) + G_3^2(T) - 2TG_2(0)G_3(0) + 2TG_2(0)G_3(T) - 2G_3(0)G_3(T)\Big) =$$

Writing it in terms of $o(T)$ similarly to the fourth moment.
($G_3^2(0) + G_3^2(T) + 2TG_2(0)G_3(T) - 2G_3(0)G_3(T) = o(T)$, Where $G_3(T)T = T * o(T^{-1}) = o(1)$ and this means that indeed it is of $o(T)$):

$$= 12\Big(T^2G_2^2(0) - 2TG_2(0)G_3(0) + o(T)\Big)$$

Now with lemma 1.2:

$$\mathcal{C}_4(S_T) = \mathbb{E}(S_T^4) - 3(\mathbb{E}(S_T^2))^2 =$$

$$= 24\Big(TG_4(0) - \frac{1}{2}T^2G_3(T) + G_2^2(0)\frac{1}{2}T^2 - 3G_2(0)TG_3(0) + o(T)\Big) +$$

$$+(-12)\Big(T^2G_2^2(0) - 2TG_2(0)G_3(0) + o(T)\Big) =$$

$$= 24TG_4(0) - 12T^2G_3(T) - 48TG_2(0)G_3(0) + o(T) =$$

From lemma 3.2 it is clear that $G_3(T) \sim o(T^{\beta-3}) = o(T^{-1})$ for $(\beta > 4)$, so from this we also know that $G_3(T)T^2$ is of $o(T)$. Thus:

$$\mathcal{C}_4(S_T) = 24T\Bigg(G_4(0) - 2G_2(0)G_3(0)\Bigg) + o(T) \tag{31}$$

We can check if its in accordance with the exponential that we calculated in the earlier chapter: (31), by lemma 3.8, reduces to:

$$\mathcal{C}_4(S_t) = 24TG_4(0) - 48TG_2(0)G_3(0) + o(T) = 24T\frac{1}{\lambda^3} - 48T\frac{1}{\lambda}\frac{1}{\lambda^2} + o(T) = \frac{-24}{\lambda^3}T + o(T)$$

Which, in fact coincides with (18)

3.4 Polynomial Case ($\beta = 4$)

In this section we are going to look closer at the case $\beta = 4$. We basically do the exact same calculations that were done in the previous subsection. However here $G_4(0) = \infty$ meaning some calculations need to be adjusted to this fact. Whenever we would get $G_4(x)$ (by integrating $G_3(x)$) we would integrate cx^{-1},so instead we use the approximation made in lemma 3.2, and work with that. This will be much easier than going over the steps again.

The following application of lemma 3.2 ($n = 3, \beta = 4$) will be used as our main tool:

$$\int_{t=0}^{T} G_3(t)dt = \int_{t=1}^{T} \left[\frac{a}{\mu \prod_{i=1}^{n}(\beta - i)} t^{-(\beta-n)} + \mathcal{O}(t^{-(\beta-n+1)}) \right] dt + \mathcal{O}(1) =$$

$$= \int_{t=1}^{T} \left[\frac{a}{6\mu} t^{-1} + \mathcal{O}(t^{-2}) \right] dt + \mathcal{O}(1) =$$

$$= \frac{a}{6\mu} \ln(T) + \mathcal{O}(1). \tag{32}$$

This is a valid approximation, because we only look at the results when T is big. In fact our approximation formula in lemma 3.2 only works as $T \to \infty$ Also there is some rule for the early behavior (e.g $[0, 1]$ interval), where this integral $G_3(t)$ of L has to be finite. We can assume that we integrate from 1 to T since only the tail distribution is known to us and the remaining term is $\mathcal{O}(1)$.

First we have already seen in the previous subsection, equation (20), that

$$\frac{1}{4!}\mathbb{E}(S_T^4) = \int_{t_4=0}^{T} \int_{t_3=0}^{t_4} \int_{t_2=0}^{t_3} \int_{t_1=0}^{t_2} \mathbb{P}(D) dt_1 dt_2 dt_3 dt_4 + \int_{t_4=0}^{T} \int_{t_3=0}^{t_4} \int_{t_2=0}^{t_3} \int_{t_1=0}^{t_2} \mathbb{P}(C \cap B) dt_1 dt_2 dt_3 dt_4$$

So here we also have to determine these 2 integrals however, we salvage as much as possible from the thought process of the ($\beta > 4$) case.

The case of $\mathbb{P}(D)$:

Here we use that equation (21) still holds:

$$\int_{t_4=0}^{T}\int_{t_3=0}^{t_4}\int_{t_2=0}^{t_3}\int_{t_1=0}^{t_2}\mathbb{P}(D)dt_1dt_2dt_3dt_4 = \int_{t_4=0}^{T}\left(-\frac{1}{2}G_2(t_4)t_4^2 - G_3(t_4)t_4 + \int_{v=0}^{t_4}G_3(v)dv\right)dt_4 =$$

with the help of (32)

$$= \int_{t_4=0}^{T}\left(-\frac{1}{2}G_2(t_4)t_4^2 - G_3(t_4)t_4\right)dt_4 + \int_{t_4=1}^{T}\left(\frac{a}{6\mu}ln(t_4) + \mathcal{O}(1)\right)dt_4 + \mathcal{O}(1) =: U$$

Where we have used that

$$\int_{t_4=0}^{1}\left(\frac{a}{6\mu}\ln(t_4) + \mathcal{O}(1)\right) = \mathcal{O}(1).$$

Lets denote this integral by U and do some smaller calculations. Now it helps to know that:

$$\int_{t_4=0}^{T} -\frac{1}{2}G_2(t_4)t_4^2dt_4 = \left[\frac{t_4^2}{2}G_3(t_4)\right]_{t_4=0}^{T} + \int_{t_4=0}^{T} -t_4G_3(t_4)dt_4 = \frac{T^2}{2}G_3(T) + \int_{t_4=0}^{T} -t_4G_3(t_4)dt_4$$

And also we can calculate the following with the help of lemma 3.2 :

$$\int_{t_4=0}^{T} -t_4G_3(t_4)dt_4 = \int_{t_4=1}^{T} -t_4\left[\frac{a}{6\mu}t_4^{-1} + \mathcal{O}(t_4^{-2})\right]dt_4 + \mathcal{O}(1) = -\frac{a}{6\mu}T + \mathcal{O}(\ln(T))$$

So now we can calculate the integrals in U:

$$U = \frac{T^2}{2}G_3(T) + 2\int_{t_4=0}^{T} -t_4G_3(t_4)dt_4 + \int_{t_4=0}^{T}\left(\frac{a}{6\mu}\ln(T) + \mathcal{O}(1)\right)dt_4 =$$

$$= \frac{T^2}{2}G_3(T) - 2\frac{a}{6\mu}T + \mathcal{O}(1) + \frac{a}{6\mu}T\ln(T) - \frac{a}{6\mu}T + \mathcal{O}(T)$$

From the fact $G_3(T) = \mathcal{O}(T^{-1})$ follows that $T^2G_3(T) = \mathcal{O}(T)$. Thus:

$$U = \frac{a}{6\mu}T\ln(T) + \mathcal{O}(T) \tag{33}$$

The case of $\mathbb{P}(C \cap B)$:

Almost everything is the same as in the case for $\beta > 4$. In particular, the decomposition (24) applies. Also, to I_1, the argument until (25) remains unchanged. That is:

$$I_1 = G_2^2(0)\frac{1}{2}T^2 - G_2(0)\int\limits_{t_4=0}^{T} t_4 G_2(t_4)dt_4 +$$

$$+G_2(0)\int\limits_{t_4=0}^{T} t_4 G_2(t_4)dt_4 - G_2(0)\int\limits_{t_4=0}^{T} [G_3(0) - G_3(t_4)]dt_4 =$$

$$= G_2^2(0)\frac{1}{2}T^2 - G_2(0)\int\limits_{t_4=0}^{T} [G_3(0) - G_3(t_4)]dt_4$$

$$= G_2^2(0)\frac{1}{2}T^2 - TG_2(0)G_3(0) + G_2(0)\int\limits_{t_4=0}^{T} G_3(t_4)dt_4 =$$

Now with the help of (32)

$$I_1 = G_2^2(0)\frac{1}{2}T^2 - TG_2(0)G_3(0) + G_2(0)[\frac{a}{6\mu}\ln(T) + \mathcal{O}(1)] = G_2^2(0)\frac{1}{2}T^2 + \mathcal{O}(T)$$

.

The integral of I_2 remains the same, that is $\in \mathcal{O}(T)$. However for the negligibility of $2I_3 + II_3$ we need to alter a tiny bit of the argument. Basically everything is the same, except equation (28). We can't write $G_4(0) - G_4(T)$ as these don't exist but rather use (32)

$$I_3 = \int\limits_{t_4=0}^{T} \Big[G_2(0)G_3(t_4) - G_2(t_4)G_3(0)\Big]dt_4 + \int\limits_{t_4=0}^{T}\int\limits_{t_3=0}^{t_4} G_2(t_4 - t_3)G_2(t_3)dt_3dt_4 =$$

$$= G_2(0)\bigg(\frac{a}{6\mu}\ln(T) + \mathcal{O}(1)\bigg) - G_3(0)\bigg(G_3(0) - G_3(T)\bigg) + \int\limits_{t_4=0}^{T}\int\limits_{t_3=0}^{t_4} G_2(t_4 - t_3)G_2(t_3)dt_3dt_4 =$$

Also during the proof that this last double integral is of $o(T)$ we didn't use that $\beta > 4$ so with that in mind:

$$= G_2(0)\mathcal{O}(T) - G_3(0)\mathcal{O}(T) + o(T) = \mathcal{O}(T)$$

Now using (24) we know that

$$\int_{t_4=0}^{T}\int_{t_3=0}^{t_4}\int_{t_2=0}^{t_3}\int_{t_1=0}^{t_2}\mathbb{P}(C\cap B)dt_1dt_2dt_3dt_4 = I_1 - 2I_2 + 2I_3 + II_3$$

$$= G_2^2(0)\frac{1}{2}T^2 + \mathcal{O}(T) \tag{34}$$

So now we know with the help of (20), (33) and (34):

$$\mathbb{E}(S_T^4) = 24\bigg(G_2^2(0)\frac{1}{2}T^2 + \frac{a}{6\mu}T\ln(T) + \mathcal{O}(T)\bigg) = 12G_2^2(0)T^2 + \frac{4a}{\mu}T\ln(T) + \mathcal{O}(T)$$

Now as for the second moment, it is exactly like in the case of $\beta > 4$:

$$3(\mathbb{E}(S_T^2))^2 = 3\Big[2\left(TG_2(0) - G_3(0) + G_3(T)\right)\Big]^2 =$$

$$= 12\Big(T^2G_2^2(0) + G_3^2(0) + G_3^2(T) - 2TG_2(0)G_3(0) + 2TG_2(0)G_3(T) - 2G_3(0)G_3(T)\Big) =$$

$$= 12T^2G_2^2(0) + \mathcal{O}(T)$$

So using lemma 1.2 and the two equations above, we get that:

$$\mathcal{C}_4(S_T) = \mathbb{E}(S_T^4) - 3(\mathbb{E}(S_T^2))^2 = 12G_2^2(0)T^2 + \frac{4a}{\mu}T\ln(T) + \mathcal{O}(T) - 12T^2G_2^2(0) + \mathcal{O}(T)$$

$$\mathcal{C}_4(S_T) = \frac{4a}{\mu}T\ln(T) + \mathcal{O}(T) \tag{35}$$

4 Summary and Outlook

In this thesis we studied a random walk S_T generated by a renewal process with its interarrival times having a tail distribution $\sim ax^{-\beta}$ (with $\beta \geq 4$). Our goal was to compare these results to the CLT expansion of the i.i.d. sum S_n, as these processes are strongly related. Below we summarize our results.

$\underline{\beta > 4}$:

In this case, the fourth moment and thus the fourth cumulant of L exists. As for S_n, the fourth cumulant appears as the coefficient of the leading error term $t^4\frac{1}{n}$ in the Taylor expansion of the logarithmic characteristic function $\ln(\psi_{S_n/\sqrt{n}}(t))$ (Which is $\mathcal{C}_4(X)/24$). Our goal was to compare this with the fourth cumulant of S_T (calculated in (31))

However it is important to note that this analogy between Renewal processes S_T and Characteristic function (which yields $S_n/\sqrt{n}$) has its limitations. It is correct in the case for $\beta = 2$, for the second cumulant (exactly because of the Central Limit Theorem). However, especially for $\beta > 4$, it is not a completely fair comparison as L and X have different cumulants. In the Renewal process part we mainly use properties of L and in the I.I.D. section X is what we actually use.

I provide some examples to better grasp the lack of analogy:

Gaussian example : Let X be the classic normal Gaussian distribution with mean 0 and variance 1, now this implies that L = |X|. As far as the i.i.d. part is concerned , here $C_4(S_n/\sqrt{n}) = 0$ as $S_n/\sqrt{n} \sim \mathcal{N}(0,1)$, whose every cumulant, except the variance, including the 4th one vanishes. However, for the renewal process part $G_n(0)$'s could be calculated as the absolute moments of the standard Gaussian. Using (31) we can determine the 4th cumulant that way, which in fact won't be 0. By lemma 3.3 (also here $\mu = \mathbb{E}(L) = \sqrt{2/\pi}$ and $\mathbb{E}(L^2) = 1$)

$$G_n(0) = \frac{1}{n!\mu}\mathbb{E}(L^n) = \frac{\sqrt{\pi}}{n!\sqrt{2}}\mathbb{E}(L^n)$$

Using recursion (with $f_L(x) = \sqrt{\frac{2}{\pi}}e^{-x^2/2}$ as density of L) for $k > 2$ and $k \in \mathbb{Z}$:

$$\mathbb{E}(L^k) = \int_0^\infty x^k f_L(x)dx = \int_0^\infty x^k \sqrt{\frac{2}{\pi}}e^{-x^2/2}dx = \sqrt{\frac{2}{\pi}}\int_0^\infty x^{k-1}xe^{-x^2/2}dx =$$

with integration by parts:

$$= (k-1)\sqrt{\frac{2}{\pi}}\int_0^\infty x^{k-2}e^{-x^2/2}dx = (k-1)\int_0^\infty x^{k-2}\sqrt{\frac{2}{\pi}}e^{-x^2/2}dx = (k-1)\cdot\mathbb{E}(L^{k-2})$$

So this means that :

$$\mathbb{E}(L^3) = 2\mathbb{E}(L) = 2\sqrt{2/\pi} \quad and \quad \mathbb{E}(L^4) = 3\mathbb{E}(L^2) = 3$$

- $\underline{G_2(0)}$:

$$G_2(0) = \frac{\sqrt{\pi}}{n!\sqrt{2}}\mathbb{E}(L^n) = \frac{\sqrt{\pi}}{2\sqrt{2}}\mathbb{E}(L^2) = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

- $\underline{G_3(0)}$:

$$G_3(0) = \frac{\sqrt{\pi}}{n!\sqrt{2}}\mathbb{E}(L^n) = \frac{\sqrt{\pi}}{6\sqrt{2}}\mathbb{E}(L^3) = \frac{\sqrt{\pi}}{6\sqrt{2}}2\sqrt{\frac{2}{\pi}} = \frac{1}{3}$$

- $\underline{G_4(0)}$:

$$G_4(0) = \frac{\sqrt{\pi}}{n!\sqrt{2}}\mathbb{E}(L^n) = \frac{\sqrt{\pi}}{24\sqrt{2}}\mathbb{E}(L^4) = \frac{\sqrt{\pi}}{24\sqrt{2}}3 = \frac{\sqrt{\pi}}{8\sqrt{2}}$$

Now according to (31):

$$\mathcal{C}_4(S_T)/T = 24\Big(G_4(0) - 2G_2(0)G_3(0)\Big) = 24\left(\frac{\sqrt{\pi}}{8\sqrt{2}} - 2\frac{\sqrt{\pi}}{2\sqrt{2}}\frac{1}{3}\right) \neq 0$$

Which is not 0 , (actually negative). This demonstrates that the analogy between the random walk and the i.i.d. sum is not so clear cut (so their cumulants are not necessarily equal).

Exponential example : It was calculated in (18) that the Renewal process yields a negative cumulant for exponential S_T/T, $-\frac{24}{\lambda^3}$ to be exact. However, the exponential distribution has a positive fourth cumulant $\frac{6}{\lambda^4}$ (see [11]). Moreover the symmetrized X also has a positive 4th cumulant, which can be easily computed by applying lemma 1.2:

$$\mathcal{C}_4(X) = \mathbb{E}(X^4) - 3(\mathbb{E}(X^2))^2 = \mathbb{E}(L^4) - 3(\mathbb{E}(L^2))^2 = \frac{24}{\lambda^4} - 3\left(\frac{2}{\lambda^2}\right)^2 = \frac{12}{\lambda^4}$$

$\underline{\beta = 4}$:

On the other hand, the 4th cumulant in $\beta = 4$ case is somewhat analogous to the 2nd moment in the $\beta = 2$ case, in the sense that the tail behavior is dominant both in the expansion of the CLT (the coefficient of the main error term $\frac{\ln(n)}{n}$) and in the scaling of the 4th cumulant of S_T (the coefficient of the term $T\ln(T)$). It is worth noting that with $\beta = 4$, $\mathcal{C}_4(S_n/\sqrt{n}) = \infty$ as the 4th moment of X is non-existent. But the coefficient $t^4\frac{\ln(n)}{n}$ in the expansion of $\ln(\psi_{S_n/\sqrt{n}}(t))$ can be interpreted as a generalized 4th cumulant (denoted as $\mathcal{C}_4^{gen}(X)$).

One observation we want to make is to compare the results of these generalized 4th cumulants. The cumulant that comes from the CLT for S_n convergence is calculated in (13) and cumulant from the asymptotics of the moments are calculated in (35). The i.i.d. result is that $\mathcal{C}_4^{gen}(S_n/\sqrt{n}) = 24\frac{a}{12} = 2a$, whereas the results from Renewal processes are

$\lim_{T\to\infty} \mathcal{C}_4(S_T)/(T\ln(T)) = \frac{4a}{\mu}$. So interestingly here also a doubling effect seems to appear.

What is next ? : An interesting observation (That happens in the case of Gaussian and exponential as well) is that the Renewal process calculations seem to provide negative cumulants in most cases. However the understanding of this property requires further insight.

So the main conclusion is, that this definitely is still an interesting area to investigate, but more research and calculations are needed in order to state rigorous theorems.

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