Limit/large dev. thms. HW assignment 1. Due Wednesday, Feb. 25 at 10.15am

Note: Each of the 3 questions is worth 10 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises with a horizontal line. Highlight the final answer. If you submit your homework electronically, pdf format is preferred (and please also hand it on paper at your earliest convenience)

1. (a) Express the logarithmic moment generating function (see page 7 of scanned lecture notes) of $aX+b$ in terms of the logarithmic moment generating function of X.

   (b) Let X and Y denote independent random variables. Express the logarithmic moment generating function of $X+Y$ in terms of the logarithmic moment generating functions of X and Y.

   (c) Let $X_1, X_2, \dots$ denote i.i.d. random variables and let N denote a non-negative integer-valued random variable, which is independent from $X_1, X_2, \dots$ Let

   $$Y = X_1 + \dots X_N.$$

   Denote by $\widehat{I}$ the log. mom. gen. function of X_i and denote by $\widehat{J}$ the log. mom. gen. function of N. Show that the log. mom. gen. function of Y is $\widehat{J} \circ \widehat{I}$ (i.e., the composition of $\widehat{J}$ and $\widehat{I}$).

2. Let $Y \sim \mathrm{POI}(10000)$ (Poisson distribution with parameter 10000). The goal of this exercise is to estimate the number of zero digits (after the decimal point) before the first non-zero digit in the decimal expansion of the probability $\mathbb{P}(Y \geq 27182)$. *Note:* $27182 \approx e \cdot 10^4$.

   You will give an upper bound and a lower bound using different methods.

   (a) Calculate the logarithmic moment generating function $\widehat{I}(\lambda)$ of the $\mathrm{POI}(\mu)$ distribution (see page 7 of the scanned lecture notes) and calculate its Legendre transform $I(x)$ (page 9 of scanned).

   (b) In order to give an upper bound on $\mathbb{P}(Y \geq 27182)$, use the *exponential Chebyshev's inequality* (i.e., the method that we used on the top of page 8 of the scanned lecture notes).

   (c) In order to give a lower bound on $\mathbb{P}(Y \geq 27182)$, estimate $\mathbb{P}(Y = 27182)$ using the crude version of Stirling's formula (page 3 of scanned).

   (d) Based on the above calculations, what is the approximate number of zero digits (after the decimal point) before the first non-zero digit in the decimal expansion of the probability $\mathbb{P}(Y \geq 27182)$?

3. *Laplace's principle.* Let $-\infty \leq a < b \leq +\infty$ and let $J : (a,b) \to \mathbb{R}$ denote a continuous function. Let us also assume that there is $x^* \in (a,b)$ for which $J(x^*) = \min_{x \in (a,b)} J(x)$ and that $\int_a^b e^{-J(x)}\,\mathrm{d}x < +\infty$. Prove that

   $$\lim_{n\to\infty} -\frac{1}{n} \ln \left(\int_a^b e^{-nJ(x)}\,\mathrm{d}x \right) = J(x^*).$$

   Hint: Prove the liminf bound and the limsup bound separately. Also, follow the advice of Terry Tao and „give yourself an epsilon of room": it is enough to show that for any $\varepsilon > 0$ we have

   $$\liminf_{n\to\infty} \frac{1}{n} \ln \left(\int_a^b e^{-nJ(x)}\,\mathrm{d}x \right) \geq -J(x^*) - \varepsilon, \qquad \limsup_{n\to\infty} \frac{1}{n} \ln \left(\int_a^b e^{-nJ(x)}\,\mathrm{d}x \right) \leq -J(x^*) + \varepsilon.$$