

Limit/large dev. thms. HW assignment 3. Due Wednesday, March 11 at 10.15am

Note: Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises with a horizontal line. Highlight the final answer. If you submit your homework electronically, pdf format is preferred.

1. (10 marks)

- (a) Let I denote the Legendre transform of the logarithmic moment generating function of X . Let $Y := X_1 + X_2$, where X_1 and X_2 are i.i.d. copies of X . Find the Legendre transform of the logarithmic moment generating function of Y .
- (b) Let I denote the Legendre transform of the logarithmic moment generating function of X . Let $Y := aX + b$ (where $a, b \in \mathbb{R}$). Find the Legendre transform of the log. mom. gen. function of Y .
- (c) Let Y_1, Y_2, \dots denote i.i.d. integer-valued random variables with distribution

$$\mathbb{P}(Y_i = -2k) = 2^{-(k+1)}, \quad k = 0, 1, 2, \dots$$

Find $\lim_{n \rightarrow \infty} \frac{1}{n} \ln(\mathbb{P}(Y_1 + \dots + Y_n \leq nx))$ for any $x \in \mathbb{R}$.

- (d) Let Y_1, Y_2, \dots denote i.i.d. integer-valued random variables with distribution

$$\mathbb{P}(Y_i = -1) = 1/4, \quad \mathbb{P}(Y_i = 0) = 1/2, \quad \mathbb{P}(Y_i = 1) = 1/4.$$

Find $\lim_{n \rightarrow \infty} \frac{1}{n} \ln(\mathbb{P}(Y_1 + \dots + Y_n \leq nx))$ for any $x \in \mathbb{R}$.

Hint: For (c) and (d), use (a) and (b), Cramér's theorem and the explicit formulas for the large deviation rate functions of famous random variables that we have learnt in class.

2. (15 marks) Let X_1, X_2, \dots denote i.i.d. random variables with $\text{EXP}(\lambda)$ distribution, i.e., the density function of X_i is $f(x) = \lambda e^{-\lambda x} \mathbb{1}[x \geq 0]$. Let $S_n = X_1 + \dots + X_n$.

- (a) Use induction on n to show that the density function of S_n is

$$f_n(x) = \lambda^n e^{-\lambda x} \frac{x^{n-1}}{(n-1)!} \mathbb{1}[x \geq 0].$$

Hint: Use the convolution formula stated on page 20 of the scanned lecture notes.

- (b) Calculate the logarithmic moment generating function $\mu \mapsto \widehat{I}(\mu)$ of X_i . For which values of μ do we have $\widehat{I}(\mu) < +\infty$?
- (c) Calculate the Legendre transform $I(x)$ of $\widehat{I}(\mu)$. For which values of x do we have $I(x) < +\infty$?
- (d) Give a formula for $\lim_{n \rightarrow \infty} \frac{1}{n} \ln(\mathbb{P}(S_n/n \geq x))$ for any $x \geq 1/\lambda$ using Cramér's theorem (see page 21 of scanned lecture notes).
- (e) Calculate $\lim_{n \rightarrow \infty} \frac{1}{n} \ln(\mathbb{P}(S_n/n \geq x))$ directly using the formula for the density function f_n of S_n .

Hint: Use Laplace's principle (similarly to page 15 of the scanned lecture notes) and the crude Stirling formula (see page 3 of scanned):

$$n^n e^{1-n} \leq n! \leq (n+1)^{n+1} e^{-n}$$

3. (5 marks) Let X_n denote an optimistic geometric random variable with success probability $p = 1/n$. Show that $X_n/\mathbb{E}(X_n)$ converges in distribution as $n \rightarrow \infty$ and identify the limiting distribution.

Hint: You need to find the cumulative distribution function (c.d.f.) F_n of $X_n/\mathbb{E}(X_n)$, i.e., you need to calculate $F_n(x) := \mathbb{P}(X_n/\mathbb{E}(X_n) \leq x)$ for each $n \in \mathbb{N}$ and $x \in \mathbb{R}$. After that, you need to show that there is a c.d.f. F such that F_n converges pointwise to F at all continuity points x of F (i.e., that $\lim_{n \rightarrow \infty} F_n(x) = F(x)$ holds for all $x \in \mathbb{R}$ for which $F(x) = \lim_{y \rightarrow x_-} F(y)$). F will turn out to be the c.d.f. of a famous probability distribution.