Limit/large dev. thms. HW assignment 4. Due Wednesday, March 18 at 10.15am

Note: Each of the 3 questions is worth 10 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises with a horizontal line. Highlight the final answer. If you submit your homework electronically, pdf format is preferred.

1. *The Fréchet distribution.*

   (a) Let $U_1, U_2, \dots$ denote i.i.d. random variables with UNI$[0,1]$ distribution. Let $\beta > 0$. Let

   $$M_n = \max\{U_1^{-\beta}, \dots, U_n^{-\beta}\}.$$

   Show that M_n/n^β converges in distribution as $n \to \infty$ by determining the cumulative distribution function (c.d.f.) $F(x)$ of the limiting distribution.

   (b) Show that if Y_1 and Y_2 are i.i.d. with the above c.d.f. $F(x)$ then $(Y_1 \vee Y_2)/2^\beta$ also has c.d.f. $F(x)$. *Instruction:* Use the explicit formula for F that you have obtained in sub-exercise (a), similarly to the top of page 44 of the scanned lecture notes.

   (c) Show that if Y_1 and Y_2 are i.i.d. with the above c.d.f. $F(x)$ then $(Y_1 \vee Y_2)/2^\beta$ also has c.d.f. $F(x)$. *Instruction:* Do not use the explicit form of F, but use the limit theorem (i.e., $M_n/n^\beta \implies Y_1$) that you have obtained in sub-exercise (a), similarly to the middle of page 44 of scanned.

2. The goal of this exercise is to deduce the central limit theorem (CLT) for Poisson distribution using the CLT for the sum of i.i.d. $EXP(1)$ random variables (proved in class, see page 47-51 of scanned).

   (a) Let $F_n : \mathbb{R} \to [0,1]$ and $F : \mathbb{R} \to [0,1]$ denote c.d.f.'s. Assume that F is continuous and $F_n \Rightarrow F$. Prove that for any convergent sequence $x_n \to x$ of real numbers we have $F_n(x_n) \to F(x)$.
   Hint: Use Slutsky.

   (b) Let $X_1, X_2, \dots$ denote i.i.d. random variables with EXP(1) distribution. We can think of X_i as the waiting time between the arrivals of consecutive earthquakes. Denote by $T_n = X_1 + \dots + X_n$ the time of the n'th earthquake. We have already determined the p.d.f. of T_n in HW3.2(a). Deduce from this that the c.d.f. of T_n is

   $$F_n(t) = \mathbb{P}(T_n \le t) = 1 - \sum_{k=0}^{n-1} e^{-t} \frac{t^k}{k!}$$

   Hint: Differentiation is easier than integration...

   (c) Denote by N_t the number of earthquakes during the time interval $[0,t]$. Show that the identity $\{T_n \le t\} = \{N_t \ge n\}$ holds and deduce from sub-exercise (b) that N_t has POI(t) distribution.

   (d) Use the fact that $\frac{T_n - n}{\sqrt{n}} \Rightarrow \mathcal{N}(0,1)$ as $n \to \infty$ to deduce that $\frac{N_t - t}{\sqrt{t}} \Rightarrow \mathcal{N}(0,1)$ as $t \to \infty$.
   Hint: You will have to use $\{T_n \le t\} = \{N_t \ge n\}$ as well as the result of sub-exercise (a).

3. *Local central limit theorem for* BIN$(n, \frac{1}{2})$

   Let $X_1, X_2, \dots$ denote i.i.d. random variables, where $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = 0) = \frac{1}{2}$. Let $S_n = X_1 + \dots + X_n$, thus $S_n \sim$ BIN$(n, \frac{1}{2})$. In this exercise we write $a_n \approx b_n$ to denote that $\lim_{n\to\infty} a_n/b_n = 1$.

   (a) Use Stirling's formula to show that if $(k(n))$ is an integer-valued sequence satisfying $k(n) \to \infty$ and $n - k(n) \to \infty$ then

   $$\frac{\sqrt{n}}{2}\mathbb{P}(S_n = k(n)) \approx \frac{1}{\sqrt{2\pi}} \frac{1}{(2k(n)/n)^{k(n)+\frac{1}{2}} \cdot (2 - 2k(n)/n)^{(n-k(n))+\frac{1}{2}}}. \qquad (1)$$

   (b) Show that if $k(n) = \frac{n}{2} + \frac{\sqrt{n}}{2} z(n)$, where $(z(n))$ is a bounded real-valued sequence, then

   $$(2k(n)/n)^{k(n)+\frac{1}{2}} \cdot (2 - 2k(n)/n)^{(n-k(n))+\frac{1}{2}} \approx e^{z(n)^2/2}.$$

   (c) Prove the *local CLT* for S_n, i.e., show that for any $x \in \mathbb{R}$ we have

   $$\lim_{n\to\infty} \frac{\sqrt{n}}{2}\mathbb{P}\left(S_n = \left\lfloor \frac{n}{2} + \frac{\sqrt{n}}{2}x \right\rfloor\right) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \qquad (2)$$

   Hint: There is a sequence $z(n)$ such that $k(n) = \left\lfloor \frac{n}{2} + \frac{\sqrt{n}}{2}x \right\rfloor = \frac{n}{2} + \frac{\sqrt{n}}{2} z(n)$ for all n.