

Limit/large dev. thms. HW assignment 5. Due April 1 at 10.15am.

Note: Each of the 3 questions is worth 10 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises with a horizontal line. Highlight the final answer. If you submit your homework electronically, pdf format is preferred.

1. Let T_1 denote the first time when the one dimensional simple symmetric random walk (X_n) reaches level 1 (see page 59 of the scanned lecture notes).

(a) Show that $\mathbb{P}(T_1 > n) = \mathbb{P}(X_n = 0) + \mathbb{P}(X_n = 1)$.

Hint: Use the reflection principle (see page 58).

(b) Use the result of an earlier homework to show that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{P}(T_1 > n)}{\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{n}}} = 1.$$

(c) Show that $\mathbb{E}(T_1) = +\infty$.

2. Let $y_{n,j} \in \mathbb{R}$, $j = 1, 2, \dots, N_n$, $n = 1, 2, \dots$ and assume

$$\lim_{n \rightarrow \infty} \max_{1 \leq j \leq N_n} |y_{n,j}| = 0, \quad \sup_n \sum_{j=1}^{N_n} |y_{n,j}| < \infty, \quad \lim_{n \rightarrow \infty} \sum_{j=1}^{N_n} y_{n,j} = y.$$

Prove

$$\lim_{n \rightarrow \infty} \prod_{j=1}^{N_n} (1 + y_{n,j}) = e^y.$$

Hint: Use the first order Taylor expansion of the logarithm function: if $|y| < \frac{1}{2}$ then $|\ln(1+y) - y| \leq Cy^2$.

3. The classical birthday paradox is the fact that if we choose 23 people randomly, then with probability at least $1/2$ there will be at least two of them who celebrate their birthdays on the same day of the year. This fact can be viewed as the $n = 365$ case of the following limit theorem.

Let us fix $n \in \mathbb{N}$ and let $X_{n,j}$, $j = 1, 2, \dots$ be i.i.d. random variables uniformly distributed on $\{1, 2, \dots, n\}$. Define

$$T_n := \min\{k : \exists j < k, X_{n,j} = X_{n,k}\}.$$

In plain words: T_n is the index j when the first coincidence of the values is observed. Note that by the pigeonhole principle, we have $\mathbb{P}(T_n \leq n + 1) = 1$.

Prove that T_n/\sqrt{n} converges weakly as $n \rightarrow \infty$ and identify the limiting distribution. More specifically, prove that for any $x \geq 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\frac{T_n}{\sqrt{n}} > x\right) = e^{-x^2/2}.$$

Hint: Use the result of exercise 2.