Midterm Exam - March 25, 2026, Limit thms. of probab.

1. Let $Y_1, Y_2, \dots$ denote i.i.d. random variables with POI(2) distribution. Let $S_n = Y_1 + \dots + Y_n$.

   (a) (2 point) Calculate the formula for the logarithmic moment generating function $\widehat{I}$ of Y_i. For which values of λ do we have $\widehat{I}(\lambda) = +\infty$?

   (b) (2 points) Calculate the formula for the Legendre transform I of the function $\widehat{I}$. For which values of x do we have $I(x) = +\infty$?

   (c) (2 points) Calculate $\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{P}\left(\frac{S_n}{n} \le x\right)\right)$ for all $x \in \mathbb{R}$ using Cramér's theorem.

   (d) (2 points) In class we have learnt an identity that relates the large deviation rate function of the BER(p) distribution and the large deviation rate function GEO(p) distribution. Propose a similar identity between the large deviation rate function I of the POI(2) distribution and the large deviation rate function of another famous probability distribution. Please provide a sketch proof of the identity that you propose using Cramér's theorem (twice).

   Solution:

   (a) $Z(\lambda) = \sum_{k=0}^{\infty} e^{-2}\frac{2^k}{k!}e^{\lambda k} = e^{-2}\exp\left(2e^{\lambda}\right) = \exp\left(2(e^{\lambda}-1)\right)$, $\lambda \in \mathbb{R}$. We have $\widehat{I}(\lambda) = \ln\left(Z(\lambda)\right) = 2(e^{\lambda}-1)$. In particular, $\widehat{I}(\lambda) < +\infty$ for all $\lambda \in \mathbb{R}$.

   (b) $I(x) = \sup_{\lambda}\{\lambda x - 2(e^{\lambda}-1)\}$. If $x<0$ then $\lim_{\lambda\to-\infty}\left(\lambda x - 2(e^{\lambda}-1)\right) = +\infty$, thus $I(x) = +\infty$. If $x > 0$ then the maximum of the function $\lambda \mapsto \lambda x - 2(e^{\lambda}-1)$ is achieved when $x = 2e^{\lambda}$, thus $\lambda = \ln(x/2)$, thus $I(x) = \ln(x/2)x - x + 2$. $I(0) = 2$.

   (c) Since $I(x) \ge 0$, $I(2) = 0$ and I is decreasing on $[0,2]$, we have

   $$\lim_{n\to\infty}\frac{1}{n}\ln\left(\mathbb{P}\left(\frac{S_n}{n}\le x\right)\right) = -\inf_{y\le x} I(y) = \begin{cases} -\infty, & \text{if } x<0,\\ -\ln(x/2)x + x - 2, & \text{if } 0 \le x < 2\\ 0, & \text{if } x \ge 2.\end{cases}$$

   (d) Note that $S_n \sim \mathrm{POI}(2n)$. Let $\tau_1, \tau_2, \dots$ denote i.i.d. random variables with EXP(2) distribution. Let $S^*(k) = \tau_1 + \dots + \tau_k$. S_n is the number of earthquakes up to time n, $S^*(k)$ is the time of the k'th earthquake (cf. HW4.2). Thus $\{S_n < \lfloor nx \rfloor\} = \{S^*(\lfloor nx\rfloor) > n\}$. If $0 < x < 2$ then

   $$e^{-nI(x)} \approxeq \mathbb{P}\left(\frac{S_n}{n}\le x\right) = \mathbb{P}(S_n < \lfloor nx\rfloor) = \mathbb{P}(S^*(\lfloor nx\rfloor) > n) = \mathbb{P}\left(\frac{S^*(\lfloor nx\rfloor)}{\lfloor nx\rfloor} > \frac{n}{\lfloor nx \rfloor}\right) \approxeq e^{-nxJ(1/x)},$$

   thus $I(x) = x\cdot J(1/x)$, where J is the large deviation rate function of the EXP(2) distribution, i.e., $J(x) = 2x - \ln(2x) - 1$. One can also check by hand that $I(x) = x \cdot J(1/x)$ indeed holds.

2. Let $X_1, X_2, \dots$ denote i.i.d. random variables with p.d.f. $f(x) = \frac{\sqrt{2}}{\pi}\frac{1}{1+x^4}$, $x \in \mathbb{R}$. Let

   $$M_n := \max\{X_1, \dots, X_n\}$$

   Find the value of $\alpha \in \mathbb{R}_+$ for which M_n/n^{α} weakly converges to a non-degenerate probability distribution and find the c.d.f. F of the limiting distribution.

   Hint: Use that if $a_n \to \infty$, $|b_n| \to 0$, $a_n \cdot b_n \to c$ then $(1+b_n)^{a_n} \to e^c$.

   Solution: Let $x > 0$. Let F denote the c.d.f. of X_i, thus $F'(x) = f(x)$. The c.d.f. of M_n/n^{α} is $F(n^{\alpha}x)^n = (1+b_n)^n$, where $b_n = -\int_{n^{\alpha}x}^{\infty} f(y)\,\mathrm{d}y$. Thus we want to find $\alpha > 0$ for which $\lim_{n\to\infty} n\cdot b_n$ is a finite (but non-zero) number. We use L'Hospital's rule in $(*)$ below:

   $$\lim_{n\to\infty} n\cdot b_n = \lim_{n\to\infty}\frac{-\int_{n^{\alpha}x}^{\infty} f(y)\,\mathrm{d}y}{1/n} \overset{(*)}{=} \lim_{n\to\infty}\frac{\alpha n^{\alpha-1}x\cdot f(n^{\alpha}x)}{-1/n^2} =$$

   $$-\alpha x\frac{\sqrt{2}}{\pi}\lim_{n\to\infty}\frac{n^{\alpha+1}}{1+n^{4\alpha}x^4} \overset{(**)}{=} -\frac{1}{3}x\frac{\sqrt{2}}{\pi}\lim_{n\to\infty}\frac{n^{4/3}}{1+n^{4/3}x^4} = -\frac{1}{3}x\frac{\sqrt{2}}{\pi}\frac{1}{x^4} = -\frac{1}{3}\frac{\sqrt{2}}{\pi}\frac{1}{x^3},$$

   where in $(**)$ we realized that we must have $\alpha + 1 = 4\alpha$, thus $\alpha = 1/3$. Thus $F(n^{1/3}x)^n = (1+b_n)^n \to F(x) = \exp\left(-\frac{1}{3}\frac{\sqrt{2}}{\pi}\frac{1}{x^3}\right)$ as $n\to\infty$ for any $x > 0$. Now $F(x) \searrow 0$ as $x \searrow 0$, thus $F(0) = 0$, thus $F(n^{1/3}x)^n \to 0$ as $n \to \infty$ for any $x \le 0$. Thus $M_n/n^{1/3} \Rightarrow X$ where the c.d.f. of X is F (Fréchet distribution).