NAME: .... NEPTUN CODE: ....

Probability Theory exam, January 10th, 2023

Working time: 100 min. Only simple, non-programmable calculators are allowed.
Maximum score: 110 points, but we consider 100 points already as 100%.

The. 1. Let X be a Poisson random variable with parameter $\lambda > 0$.

(a) *(6 points)* Define the probability weight function of X. Prove the formulas for the expected value and variance of X.

(b) *(8 points)* State and prove the Poisson approximation theorem of binomial random variables.

(c) *(5 points)* We have a box with a lot of lightblubs in it; each of which is faulty with a small probability. We know that the probability of having zero faulty lightblubs is $2/3$. What is the probability that there are at most 2 faulty blubs in the box?

The. 2. Consider two random variables X, Y.

(a) *(4 points)* Define their covariance and prove that $\text{Cov}(X, Y)$ is bilinear. Name the properties of expectation when you use them.

(b) *(2 points)* Prove that $\text{Cov}(X, Y) = 0$ if X, Y are independent. Name the properties of expectation when you use them.

(c) *(7 points)* Is it also true, that X, Y are independent if $\text{Cov}(X, Y) = 0$? Why?

The. 3. Let $\lambda, \mu > 0$ and let $X \sim \text{Poi}(\lambda), Y \sim \text{Poi}(\mu)$ be independent random variables.

(a) *(4 points)* Calculate the moment generating function of X.

(b) *(4 points)* Using moment generating functions, prove that $X + Y \sim \text{Poi}(\lambda + \mu)$. Explain all steps of your proof.

(c) *(10 points)* Let $Z|X \sim \text{Bin}(X, p)$ for some constant $p \in [0, 1]$. Prove that $Z \sim \text{Poi}(p\lambda)$.

Prac. 1. *(20 points)* To celebrate Advent, Anne and Belle made tea selections for each other. Both selections consist of 24 teabags; placed onto one another in a small gift box. The two girls organized the teabags randomly, choosing each permutation with the same probability, independently of each other. There are exactly 8 black and 6 green teas in a gift box. On each day of Advent, they drink the uppermost tea in their respective box, and they only drink one tea per day.

Let X denote the number of days when they both drink black tea. Calculate $\mathbb{E}[X]$ and $\text{Var}(X)$.

BONUS: *(10 points)* Let Y denote the number of days when they both drink green tea. Calculate $\text{Cov}(X, Y)$.

Prac. 2. *(15 points)* First, we roll with a fair die and write X for the result. Then, we keep throwing two fair dice until the sum of the rolled numbers satisfy a certain condition depending on X. If X is smaller than 5, then we stop when the sum is even. Otherwise, we stop when the sum is divisible by 3. What is the probability that the sum is bigger or equal to 10 when we stop?

Prac. 3. Jack arrives to school with a delay of $X \sim \text{Uni}\,[0, 10]$ minutes. Jill arrives with a delay of $Y \sim \text{Uni}\,[0, 15]$ minutes, independently of Jack.

(a) *(5 points)* What is the probability that Jill arrives earlier than Jack?

(b) *(10 points)* What is the expectation of the difference $|X - Y|$?

$\Phi(z)$

z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000