$\boxed{\text{THE 1}}$ a) $\mathbb{P}(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}, \; k = 0, 1, 2, \dots$

$$\mathbb{E}(X) = \sum_{k=0}^{\infty} k \cdot \mathbb{P}(X = k) = \lambda \cdot \sum_{k=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{k-1}}{(k-1)!} = \lambda \cdot 1 = \lambda$$

$$\mathbb{E}(X \cdot (X-1)) = \sum_{k=0}^{\infty} k \cdot (k-1) \cdot \mathbb{P}(X = k) = \lambda^2 \cdot \sum_{k=2}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{k-2}}{(k-2)!} = \lambda^2 \cdot 1 = \lambda^2$$

$$\mathbb{E}(X^2) = \mathbb{E}(X \cdot (X-1)) + \mathbb{E}(X) = \lambda^2 + \lambda$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

---

b) THM: IF $X_n \sim \mathrm{BIN}(n, p_n)$ AND $n \cdot p_n \xrightarrow[\infty]{n} \lambda$

AND $X \sim \mathrm{POI}(\lambda)$ THEN $\forall k \in \mathbb{N}: \mathbb{P}(X_n = k) \xrightarrow[\infty]{n} \mathbb{P}(X = k)$

<u>PROOF</u>: $\mathbb{P}(X_n = k) = \binom{n}{k} \cdot p_n^k \cdot (1-p_n)^{n-k} =$

$$= \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{k!} \cdot p_n^k \cdot (1-p_n)^n \cdot (1 + \bar{\bar{o}}(1)) =$$

$$\frac{1}{k!} \prod_{i=0}^{k-1} (n-i) \cdot p_n \cdot e^{-n \cdot p_n} \cdot (1 + \bar{\bar{o}}(1)) = \frac{1}{k!} \cdot \lambda^k \cdot e^{-\lambda} \cdot (1 + \bar{\bar{o}}(1)) \rightarrow$$

$$\xrightarrow[\infty]{n} e^{-\lambda} \cdot \lambda^k \cdot \frac{1}{k!} = \mathbb{P}(X = k) \checkmark$$

BECAUSE OF b)

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c) NUMBER OF LIGHTBULBS IN BOX: $X \sim \mathrm{POI}(\lambda)$

$$\mathbb{P}(X = 0) = \frac{2}{3} \Rightarrow e^{-\lambda} = \frac{2}{3} \Rightarrow \lambda = \ln\left(\frac{3}{2}\right) \Rightarrow$$

$$\mathbb{P}(X \leq 2) = \underbrace{e^{-\lambda}}_{= \frac{2}{3}} \cdot \left(1 + \lambda + \frac{\lambda^2}{2}\right) \approx 0.9918$$

THE 2 a) $\mathrm{Cov}(X,Y) \overset{☺}{=} \mathbb{E}(XY) - \mathbb{E}(X)\cdot\mathbb{E}(Y)$

ALTERNATIVE DEF: $\mathrm{Cov}(X,Y) \overset{☹}{=} \mathbb{E}\big((X-\mathbb{E}(X))\cdot(Y-\mathbb{E}(Y))\big)$

BILINEARITY: $\mathrm{Cov}\left(\sum_{i=1}^{n} \alpha_i X_i, \sum_{j=1}^{m} \beta_j Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i \beta_j \mathrm{Cov}(X_i, Y_j)$

PROOF: W.L.O.G. WE MAY ASSUME THAT $\mathbb{E}(X_i) = \mathbb{E}(Y_j) = 0$ BECAUSE OF ☹ AND THE LINEARITY OF EXPECT. THEN:

$$\mathrm{Cov}\left(\sum_i \alpha_i X_i, \sum_j \beta_j Y_j\right) = \mathbb{E}\left(\left(\sum_i \alpha_i X_i\right)\left(\sum_j \beta_j Y_j\right)\right) =$$

$$\mathbb{E}\left(\sum_i \sum_j \alpha_i \beta_j X_i Y_j\right) \overset{LIN}{=} \sum_i \sum_j \alpha_i \beta_j \underbrace{\mathbb{E}(X_i Y_j)}_{\mathrm{Cov}(X_i, Y_j)}$$

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b) IF X AND Y ARE INDEP. THEN $\mathbb{E}(XY) = \mathbb{E}(X)\cdot\mathbb{E}(Y)$, THUS $\mathrm{Cov}(X,Y) \overset{☺}{=} 0$

---

c) NOT TRUE. HERE IS A COUNTEREXAMPLE:
LET U AND V DENOTE I.I.D. DIE ROLLS.
LET $X := U+V$ AND $Y := U-V$. THEN

$$\mathrm{Cov}(X,Y) \overset{BIL}{=} \mathrm{Cov}(U,U) - \mathrm{Cov}(U,V) + \mathrm{Cov}(U,V) - \mathrm{Cov}(V,V) = 0$$

BUT X AND Z, ARE NOT INDEPENDENT BECAUSE

$$\mathbb{P}(X=12, Y=5) = 0 \neq \left(\frac{1}{36}\right)^2 = \mathbb{P}(X=12)\cdot\mathbb{P}(Y=5)$$

THE 3

LAW OF THE UNCONSCIOUS STATISTICIAN

a) $M_X(t) = \mathbb{E}\left(e^{t\cdot X}\right) = \sum_{x=0}^{\infty} e^{t\cdot x}\cdot e^{-\lambda}\cdot\frac{\lambda^x}{x!} =$

$= e^{-\lambda}\cdot\sum_{x=0}^{\infty}\frac{(e^t\cdot\lambda)^x}{x!} = e^{-\lambda}\cdot\exp(e^t\cdot\lambda) = \exp(\lambda\cdot(e^t-1))$

---

b) LET $U := X + Y$

INDEP.

$M_U(t) = \mathbb{E}\left(e^{t\cdot U}\right) = \mathbb{E}\left(e^{t\cdot X}\cdot e^{t\cdot Y}\right) = \mathbb{E}\left(e^{t\cdot X}\right)\cdot\mathbb{E}\left(e^{t\cdot Y}\right) =$

$= \exp(\lambda\cdot(e^t-1))\cdot\exp(\mu\cdot(e^t-1)) = \exp((\lambda+\mu)\cdot(e^t-1))$

THIS IS THE M.G.F. OF $POI(\lambda+\mu)$, THUS $U \sim POI(\lambda+\mu)$, SINCE THE M.G.F. UNIQUELY IDENTIFIES THE DISTRIBUTION.

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c) IF $X = n$ THEN $Z_1 \sim BIN(n,p)$

SIMILARLY TO a)

THUS $\mathbb{E}\left(e^{t\cdot Z_1} \mid X = n\right) = \left(p\cdot e^t + (1-p)\cdot e^0\right)^n$

TOWER RULE: $\mathbb{E}\left(e^{t\cdot Z_1}\right) = \mathbb{E}\left(\left(p\cdot e^t + (1-p)\right)^X\right) =$

$\exp(\lambda\cdot(p\cdot e^t + (1-p) - 1)) = \exp(\lambda\cdot p\cdot(e^t-1))$

THIS IS THE M.G.F. OF $POI(p\cdot\lambda)$, THUS $Z_1 \sim POI(p\cdot\lambda)$ SINCE THE M.G.F. UNIQUELY IDENTIFIES THE DISTRIBUTION.

$\boxed{\text{PRAC } 1}$ $A_i := \{\text{BOTH OF THEM DRINK BLACK ON } i\text{'TH DAY}\}$

$X_i := \mathbb{1}[A_i]$, THEN $X = \sum_{i=1}^{24} X_i$

$$\mathbb{E}(X) \overset{\text{LIN}}{=} \sum_{i=1}^{24} \mathbb{E}(X_i) = 24 \cdot \mathbb{P}(A_1) = 24 \cdot \left(\frac{8}{24}\right)^2 = \frac{8}{3} = 2.\dot{6}$$

$$\text{Var}(X) = \text{Cov}(X, X) = \text{Cov}\left(\sum_{i=1}^{24} X_i, \sum_{j=1}^{24} X_j\right) = \sum_{i,j=1}^{24} \text{Cov}(X_i, X_j) =$$
$$= ☆$$

$\boxed{i=j}$: $\text{Cov}(X_i, X_i) = \text{Var}(X_i) = \frac{1}{9} \cdot \left(1 - \frac{1}{9}\right) = \frac{8}{81} = 0.0988$

$\boxed{i \neq j}$ $\text{Cov}(X_i, X_j) = \mathbb{P}(A_i \cap A_j) - \underbrace{\mathbb{P}(A_i)}_{\frac{1}{9}} \cdot \underbrace{\mathbb{P}(A_j)}_{\frac{1}{9}} = -0.002$

$\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i) \cdot \mathbb{P}(A_j | A_i) = \frac{1}{9} \cdot \left(\frac{7}{23}\right)^2$

$$☆ = 24 \cdot 0.0988 + 24 \cdot 23 \cdot (-0.002) = 1.2672 = \text{Var}(X)$$

BONUS: $B_i := \{\text{BOTH OF THEM DRINK GREEN ON } i\text{'TH DAY}\}$

$Y_i := \mathbb{1}[B_i]$, THEN $Y = \sum_{i=1}^{24} Y_i$

$$\text{Cov}(X, Y) = \text{Cov}\left(\sum_{i=1}^{24} X_i, \sum_{j=1}^{24} Y_j\right) \overset{\text{B/C}}{=} \sum_{i,j=1}^{24} \text{Cov}(X_i, Y_j) = ☺$$

$\boxed{i=j}$: $\text{Cov}(X_i, Y_i) = \underbrace{\mathbb{P}(A_i \cap B_i)}_{0} - \frac{1}{9} \cdot \frac{1}{16} = -0.0069\dot{4}$

$\boxed{i \neq j}$: $\text{Cov}(X_i, Y_j) = \underbrace{\mathbb{P}(A_i \cap B_j)}_{= \frac{1}{9} \cdot \left(\frac{6}{23}\right)^2} - \frac{1}{9} \cdot \frac{1}{16} = 0.000617$

$$☺ = 24 \cdot (-0.0069\dot{4}) + 24 \cdot 23 \cdot 0.000617 = 0.1739$$

PRAC 2 : IF U AND V ARE I.I.D. FAIR DIE ROLLS AND $W := U + V$ THEN

$\mathbb{P}(W \text{ IS EVEN}) = \frac{1}{2}$, $\mathbb{P}(W \text{ IS DIVISIBLE BY } 3) = \frac{1}{3}$

$$\mathbb{P}(W \geq 10 \mid W \text{ IS EVEN}) = \frac{\mathbb{P}(W \geq 10 \text{ AND } W \text{ IS EVEN})}{1/2}$$

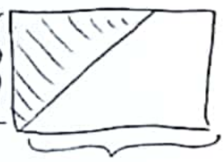
$$= \frac{\mathbb{P}(W=10) + \mathbb{P}(W=12)}{1/2} = \frac{3/36 + 1/36}{1/2} = \frac{2}{9}$$

$$\mathbb{P}(W \geq 10 \mid W \text{ IS DIVISIBLE BY } 3) = \frac{\mathbb{P}(W=12)}{1/3} = \frac{1}{12}$$

$\mathbb{P}(\text{SUM IS AT LEAST 10 WHEN WE STOP}) =$

$$\frac{2}{9} \cdot \underbrace{\mathbb{P}(X < 5)}_{\frac{2}{3}} + \frac{1}{12} \cdot \underbrace{\mathbb{P}(X \geq 5)}_{\frac{1}{3}} \approx 0.1759$$

$\boxed{\text{PRAC 3}}$ a) 10



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$$\text{PROB} = \frac{10^2/2}{10 \cdot 15} = \frac{50}{150} = \frac{1}{3}$$

LAW OF THE UNCONSCIOUS STATISTICIAN

$$\text{b)} \quad \mathbb{E}(|X - Y|) = \frac{1}{10} \cdot \frac{1}{15} \cdot \int_0^{10} \int_0^{15} |x-y| \, dy \, dx =$$

$$\frac{1}{150} \cdot \Big(\underbrace{\int_0^{10} \int_0^{x} (x-y) \, dy \, dx}_{\frac{1000}{6}} + \underbrace{\int_0^{10} \int_x^{15} (y-x) \, dy \, dx}_{= \frac{10 \cdot 15^2}{2} - \frac{1500}{2} + \frac{1000}{6}} \Big) = 4.7\dot{2}$$

$$\int_0^{10}\int_0^{x}(x-y)\,dy\,dx=\int_0^{10}\int_0^{x}y\,dy\,dx=\int_0^{10}\frac{x^2}{2}\,dx=\frac{1}{2}\cdot\frac{10^3}{3}=\frac{1000}{6}$$

$$\int_0^{10}\int_x^{15}(y-x)\,dy\,dx=\int_0^{10}\left[\frac{y^2}{2}-xy\right]_x^{15}dx=$$

$$=\int_0^{10}\left(\left(\frac{15^2}{2}-x\cdot 15\right)-\left(\frac{x^2}{2}-x^2\right)\right)dx=$$

$$=\int_0^{10}\left(\frac{15^2}{2}-15x+\frac{x^2}{2}\right)dx=10\cdot\frac{15^2}{2}-15\cdot\int_0^{10}x\,dx+\frac{1}{2}\,\frac{10^3}{3}$$

$$=\frac{10\cdot 15^2}{2}-15\cdot\frac{10^2}{2}+\frac{1000}{6}$$