$\boxed{\text{ELM 1.}}$ a) $\mathbb{P}(X=k) = e^{-\lambda}\cdot\frac{\lambda^k}{k!}$, $k=0,1,2,\dots$

$$\mathbb{E}(X) = \sum_{k=0}^{\infty} k\cdot\mathbb{P}(X=k) = \lambda\cdot\sum_{k=1}^{\infty} e^{-\lambda}\cdot\frac{\lambda^{k-1}}{(k-1)!} = \lambda\cdot 1 = \lambda$$

$$\mathbb{E}(X\cdot(X-1)) = \sum_{k=0}^{\infty} k\cdot(k-1)\cdot\mathbb{P}(X=k) = \lambda^2\cdot\sum_{k=2}^{\infty} e^{-\lambda}\cdot\frac{\lambda^{k-2}}{(k-2)!} = \lambda^2\cdot 1 = \lambda^2$$

$$\mathbb{E}(X^2) = \mathbb{E}(X\cdot(X-1)) + \mathbb{E}(X) = \lambda^2+\lambda$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \lambda^2+\lambda-\lambda^2 = \lambda$$

---

b) TÉTEL: HA $X_n \sim BIN(n, p_n)$, $n\cdot p_n \xrightarrow[n\to\infty]{} \lambda$ ÉS $X \sim POI(\lambda)$, AKKOR $\mathbb{P}(X_n=k) \xrightarrow[n\to\infty]{} \mathbb{P}(X=k)$

BIZ: $\mathbb{P}(X_n=k) = \binom{n}{k}\cdot p_n^k\cdot(1-p_n)^{n-k} =$

$$= \frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{k!}\cdot p_n^k\cdot(1-p_n)^n\cdot(1+\bar{\bar{o}}(1)) =$$

$$\frac{1}{k!}\prod_{i=0}^{k-1}(n-i)\cdot p_n\cdot e^{-n\cdot p_n}\cdot(1+\bar{\bar{o}}(1)) = \frac{1}{k!}\cdot\lambda^k\cdot e^{-\lambda}\cdot(1+\bar{\bar{o}}(1)) \to$$

$$\xrightarrow[n\to\infty]{} e^{-\lambda}\cdot\lambda^k\cdot\frac{1}{k!} = \mathbb{P}(X=k) \checkmark$$

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b) MIATT

c) DOBOZBAN LEVŐ KÖRTÉK SZÁMA: $X \sim POI(\lambda)$

$\mathbb{P}(X\geq 1) = \frac{2}{3} \Rightarrow \mathbb{P}(X=0) = \frac{1}{3} \Rightarrow e^{-\lambda} = \frac{1}{3} \Rightarrow$

$\Rightarrow \lambda = \ln(3) \Rightarrow \mathbb{P}(X\leq 2) = \underbrace{e^{-\lambda}}_{=\frac{1}{3}}\cdot\left(1+\lambda+\frac{\lambda^2}{2}\right) = 0.90069$

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ELM 2. a) $Cov(X,Y) \overset{☺}{=} \mathbb{E}(XY) - \mathbb{E}(X)\cdot\mathbb{E}(Y)$

MÁSIK DEF: $Cov(X,Y) \overset{☹}{=} \mathbb{E}((X-\mathbb{E}(X))\cdot(Y-\mathbb{E}(Y)))$

BILINEARITÁS: $Cov\left(\sum_{i=1}^{n} \alpha_i X_i, \sum_{j=1}^{m} \beta_j Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i \beta_j Cov(X_i, Y_j)$

BIZ: ☹ MIATT ÉS A VÁRHATÓ ÉRTÉK LINEARITÁSA MIATT FELTEHETJÜK, HOGY $\mathbb{E}(X_i)=0$ ÉS $\mathbb{E}(Y_j)=0$. EKKOR:

$$Cov\left(\sum_i \alpha_i X_i, \sum_j \beta_j Y_j\right) = \mathbb{E}\left(\left(\sum_i \alpha_i X_i\right)\cdot\left(\sum_j \beta_j Y_j\right)\right) =$$

$$\mathbb{E}\left(\sum_i \sum_j \alpha_i \beta_j X_i Y_j\right) \overset{LIN.}{=} \sum_i \sum_j \alpha_i \beta_j \underbrace{\mathbb{E}(X_i Y_j)}_{Cov(X_i, Y_j)}$$

---

b) HA X ÉS Y FÜGGETLENEK, AKKOR $\mathbb{E}(XY) = \mathbb{E}(X)\cdot\mathbb{E}(Y)$, ÍGY $Cov(X,Y) \overset{☺}{=} 0$

---

c) NEM IGAZ. ELLENPÉLDA: LEGYENEK U ÉS V F.A.E. KOCKADOBÁSOK. LEGYEN $X := U+V$ ÉS $Y := U-V$. EKKOR

$$Cov(X,Y) = Cov(U+V, U-V) \overset{BIL.}{=}$$

$$Cov(U,U) - Cov(U,V) + Cov(V,U) - Cov(V,V) =$$

$Var(U) - Var(V) = 0$ VISZONT X ÉS Y NEM FÜGGETLENEK, HISZ $\mathbb{P}(X=12, Y=5) = 0$, VISZONT $\underbrace{\mathbb{P}(X=12)}_{1/36} \cdot \underbrace{\mathbb{P}(Y=5)}_{1/36} > 0$

2. OLDAL

$\boxed{\text{ELM 3.}}$ a) $\mathcal{X} \sim N(0,1)$ $\quad \tilde{M}(t) := \mathbb{E}\left(e^{t\mathcal{X}}\right) \underset{\uparrow}{=}$ LAW OF UNCONSCIOUS STAT.

$$= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}\, dx = e^{t^2/2} \cdot \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-(x-t)^2/2}\, dx}_{1} = e^{t^2/2}$$

---

b) LEGYEN $\mathcal{Y}_n := S_n/\sqrt{n}$. $M_n(t) := \mathbb{E}\left(e^{t \cdot \mathcal{Y}_n}\right)$

ELÉG BELÁTNI, HOGY $\forall t \in \mathbb{R}$ $M_n(t) \xrightarrow[\infty]{n} \tilde{M}(t) = e^{t^2/2}$,

MERT AZ ELŐADÁSRÓL TUDJUK, HOGY EBBŐL KÖVETKEZIK

$$\mathbb{P}(\mathcal{Y}_n \le x) \xrightarrow[\infty]{n} \mathbb{P}(\mathcal{X} \le x) = \Phi(x)$$

$M_n(t) = \mathbb{E}\left(e^{t/\sqrt{n} \cdot (\mathcal{X}_1 + \ldots + \mathcal{X}_n)}\right) = M\left(\frac{t}{\sqrt{n}}\right)^n$, HISZ

FÜGGETLEN VAL. VÁLTOZÓK ÖSSZEGÉNEK MOM. GEN. FÜGGVÉNYE EGYENLŐ AZ ÖSSZEADANDÓK MOM. GEN. FÜGGVÉNYEINEK A SZORZATÁVAL.

ELÉG BELÁTNI: $\ln(M_n(t)) \xrightarrow[\infty]{n} t^2/2$

$$M(t) \underset{\text{MÁSODRENDŰ TAYLOR}}{=} 1 + \underbrace{M'(0)}_{\mathbb{E}(\mathcal{X}_i)=0} \cdot t + \frac{1}{2}\underbrace{M''(0)}_{\mathbb{E}(\mathcal{X}_i^2)=1} \cdot t^2 + \bar{\bar{o}}(t^2) = 1 + \frac{1}{2}t^2 + \bar{\bar{o}}(t^2)$$

$\ln(M(t)) = \frac{1}{2}t^2 + \bar{\bar{o}}(t^2)$, $\quad \ln(M_n(t)) = n \cdot \ln\left(M\left(\frac{t}{\sqrt{n}}\right)\right) =$

$$n \cdot \ln\left(1 + \frac{1}{2} \cdot \left(\frac{t}{\sqrt{n}}\right)^2 + \bar{\bar{o}}\left(\left(\frac{t}{\sqrt{n}}\right)^2\right)\right) = n \cdot \left(\frac{1}{2} \cdot \frac{t^2}{n} + \bar{\bar{o}}\left(\frac{t^2}{n}\right)\right) \xrightarrow[\infty]{n} \frac{t^2}{2}$$

ELSŐRENDŰ TAYLOR $\rightarrow$ $\ln(1+x) = x + \bar{\bar{o}}(x)$

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GYAK 1 $A_i := \{$ i-EDIK NAPON MINDKETTEN FEKETÉT ISZ. $\}$

$X_i := \mathbb{1}[A_i]$, EKKOR $X = \sum_{i=1}^{24} X_i$

$\mathbb{E}(X) \overset{LIN}{=} \sum_{i=1}^{24} \mathbb{E}(X_i) = 24 \cdot \mathbb{P}(A_1) = 24 \cdot \left(\frac{8}{24}\right)^2 = \frac{8}{3} = 2,\dot{6}$

$\mathrm{Var}(X) = \mathrm{Cov}(X, X) = \mathrm{Cov}\left(\sum_{i=1}^{24} X_i, \sum_{j=1}^{24} X_j\right) = \sum_{i,j=1}^{24} \mathrm{Cov}(X_i, X_j) = ⊛$

$\boxed{i=j}$: $\mathrm{Cov}(X_i, X_j) = \mathrm{Var}(X_i) = \frac{1}{9} \cdot \left(1 - \frac{1}{9}\right) = \frac{8}{81} = 0.0988$

$\boxed{i \neq j}$: $\mathrm{Cov}(X_i, X_j) = \mathbb{P}(A_i \cap A_j) - \underbrace{\mathbb{P}(A_i)}_{\frac{1}{9}} \cdot \underbrace{\mathbb{P}(A_j)}_{\frac{1}{9}} = -0.002$

$\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i) \cdot \mathbb{P}(A_j | A_i) = \frac{1}{9} \cdot \left(\frac{7}{23}\right)^2$

$⊛ = 24 \cdot 0.0988 + 24 \cdot 23 \cdot (-0.002) = 1.2672 = \mathrm{Var}(X)$

BÓNUSZ: $B_i = \{$ i-EDIK NAPON MINDKETTEN ZÖLDET ISZ. $\}$

$\mathbb{P}\left(\bigcap_{i=1}^{24} B_i^C\right) = 1 - \mathbb{P}\left(\bigcup_{i=1}^{24} B_i\right)$

$\mathbb{P}\left(\bigcup_{i=1}^{24} B_i\right) \overset{\text{SZITA}}{=} \sum_{\substack{I \subseteq [24] \\ I \neq \emptyset}} (-1)^{|I|+1} \cdot \underbrace{\mathbb{P}\left(\bigcap_{i \in I} B_i\right)}_{= \left(\frac{\binom{24-|I|}{6-|I|}}{\binom{24}{6}}\right)^2} = ☺$

$☺ = \sum_{k=1}^{6} (-1)^{k+1} \cdot \binom{24}{k} \cdot \left(\frac{\binom{24-k}{6-k}}{\binom{24}{6}}\right)^2$

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$\boxed{\text{GYAK 2}}$ a) $f_{X}(x)=\frac{1}{3}\cdot \mathbb{1}[0\le x\le 3]$

$$f_{Y|X}(y|x)=\frac{1}{3-x}\cdot \mathbb{1}[0\le y\le 3-x]$$

EGYÜTTES SŰ. FV.: $f(x,y)=f_{X}(x)\cdot f_{Y|X}(y|x)=$

$$=\frac{1}{3}\cdot\frac{1}{3-x}\cdot \mathbb{1}[0\le x\le 3,\ 0\le y\le 3-x]$$

---

b) $\mathbb{E}(Y|X)=\frac{3-X}{2}$ $\quad \mathbb{E}(Y)\overset{\text{TORONY}}{=}\mathbb{E}\left(\frac{3-X}{2}\right)=\frac{3-\mathbb{E}(X)}{2}=\frac{3}{4}$

HISZ Y EGYENLETES A $[0,3-X]$ INTERVALLUMON

HISZ X EGYENLETES A $[0,3]$ INT.

$$\mathrm{Var}(Y|X)=\frac{(3-X)^2}{12}$$

MEGJ.: HA $V:=3-X$, AKKOR $V\sim \mathrm{UNI}[0,3]$

$$\mathrm{Var}(Y)=\mathbb{E}\left(\frac{(3-X)^2}{12}\right)+\mathrm{Var}\left(\frac{3-X}{2}\right)=\text{☺}$$

FELT. SZÓR. NÉGYZET FORMULA

$$\text{☺}=\underbrace{\mathbb{E}\left(\frac{V^2}{12}\right)}_{\frac{1}{12}\cdot\left(\mathbb{E}(V)^2+\mathrm{Var}(V)\right)=\frac{1}{12}\cdot\left(\left(\frac{3}{2}\right)^2+\frac{3^2}{12}\right)=\frac{1}{4}}+\frac{1}{4}\cdot\underbrace{\mathrm{Var}(X)}_{\frac{3^2}{12}}=\frac{1}{4}+\frac{3}{16}=\frac{7}{16}=\mathrm{Var}(Y)$$

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$\boxed{\text{FYAK 3}}$ $S_{25} = X_1 + \ldots + X_{25}$

$S'_{64} = X'_1 + \ldots + X'_{64}$

$$Y := \frac{S_{25} - 25\cdot 75}{5\cdot 13} \overset{\text{C.H.T}}{\approx} N(0,1)$$

$$Y' := \frac{S'_{64} - 64\cdot 75}{8\cdot 13} \overset{\text{C.H.T.}}{\approx} N(0,1)$$

Y ÉS Y' FÜGGETLENEK

$$\mathbb{P}\left(\left|\frac{S_{25}}{25} - \frac{S'_{64}}{64}\right| \geq 2\right) =$$

$$\mathbb{P}\left(\left|\frac{5\cdot 13\cdot Y + 25\cdot 75}{25} - \frac{8\cdot 13\cdot Y' + 64\cdot 75}{64}\right| \geq 2\right) =$$

$$\mathbb{P}\left(\left|\frac{13}{5}Y - \frac{13}{8}Y'\right| \geq 2\right) = \star$$

$$Z := \frac{13}{5}Y - \frac{13}{8}Y', \quad Z \sim N\Big(0, \underbrace{\left(\tfrac{13}{5}\right)^2 + \left(\tfrac{13}{8}\right)^2}_{\boxed{\sigma^2 \approx 9.4}}\Big) \quad \boxed{\sigma = 3.066}$$

$$\star = \mathbb{P}(|Z| \geq 2) = \mathbb{P}\left(\left|\frac{Z}{\sigma}\right| \geq \frac{2}{\sigma}\right) =$$

$$= 2\cdot\left(1 - \Phi\left(\frac{2}{\sigma}\right)\right) = 2\cdot\left(1 - \Phi(0.6523)\right) =$$

$$= 2\cdot(1 - 0.7324) = 0.5352$$

6. OLDAL

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