MOMENTUMGENERÁLÓ FÜGGVÉNY:

HA $\mathbb{X}$ EGY VA. VÁ., AKKOR $\boxed{M_{\mathbb{X}}(t) := \mathbb{E}\left(e^{t\cdot\mathbb{X}}\right)}$

TÉNY: $\boxed{M'_{\mathbb{X}}(0) = \mathbb{E}(\mathbb{X})}$ $\boxed{M''_{\mathbb{X}}(0) = \mathbb{E}(\mathbb{X}^2)}$

$\boxed{12.1}$ $\mathbb{X} \sim \text{OPTGEO}(p)$: $\mathbb{P}(\mathbb{X} = k) = p\cdot(1-p)^{k-1}$

LAW OF UNCONSCIOUS STATISTICIAN

$k = 1, 2, \dots$

$$\mathbb{E}\left(e^{t\cdot\mathbb{X}}\right) = \sum_{k=1}^{\infty} e^{t\cdot k}\cdot\mathbb{P}(\mathbb{X} = k) = \sum_{k=1}^{\infty} e^{t\cdot k}\cdot p\cdot(1-p)^{k-1} =$$

$$= p\cdot e^{t}\cdot\sum_{k=1}^{\infty}\left(e^{t}\cdot(1-p)\right)^{k-1} = \begin{cases} +\infty, & \text{HA } e^{t}\cdot(1-p) \geq 1 \\ \dfrac{p\cdot e^{t}}{1-e^{t}\cdot(1-p)}, & \text{HA } e^{t}\cdot(1-p) < 1 \end{cases}$$

TEHÁT $\boxed{M(t) = \dfrac{p}{e^{-t}-(1-p)}}$, HA $\boxed{t < \ln\left(\dfrac{1}{1-p}\right)}$

$$\boxed{M'(t) = \frac{p\cdot e^{-t}}{(p+e^{-t}-1)^2}} \qquad \boxed{M''(t) = p\cdot\left(\frac{2\cdot e^{-2t}}{(p+e^{-t}-1)^3} - \frac{e^{-t}}{(p+e^{-t}-1)^2}\right)}$$

$\mathbb{E}(\mathbb{X}) = M'(0) = \dfrac{1}{p}$ $\quad$ $\text{Var}(\mathbb{X}) = M''(0) - \left(\dfrac{1}{p}\right)^2 =$

$$= p\cdot\left(\frac{2}{p^3} - \frac{1}{p^2}\right) - \frac{1}{p^2} = \frac{1}{p^2} - \frac{1}{p} = \frac{1-p}{p^2} \checkmark$$

$X \sim UNI[a, b]$: $f(x) = \frac{1}{b-a} \cdot \mathbb{1}[a < x < b]$

$$\mathbb{E}\left(e^{t \cdot X}\right) = \int_{-\infty}^{\infty} e^{t \cdot x} \cdot f(x)\, dx = \int_a^b e^{t \cdot x} \cdot \frac{1}{b-a}\, dx \underset{\boxed{\text{HA } t \neq 0}}{=}$$

$$= \frac{1}{t \cdot (b-a)} \cdot \left(e^{t \cdot b} - e^{t \cdot a}\right) = M(t)$$

$$\lim_{t \to 0} M(t) \overset{\text{L'HOSPITAL}}{=} 1, \text{ ÉS } \mathbb{E}\left(e^{0 \cdot X}\right) = 1 \checkmark$$

$$M'(t) = \ldots = \frac{(a \cdot t - 1) \cdot e^{a \cdot t} + (1 - b t) \cdot e^{b \cdot t}}{t^2 \cdot (a-b)}$$

$$\lim_{t \to 0} M'(t) \overset{\text{L'HOSPITAL}}{\underset{\text{ÉS MÉG EGYSZER L'HOSPITAL}}{=\!=}} \frac{a^2 - b^2}{2 \cdot (a-b)} = \frac{a+b}{2} \checkmark$$

<u>CENTRÁLIS</u> <u>HATÁRELOSZLÁS</u>-<u>TÉTEL</u> (C.H.T.):

HA $X_1, X_2, \ldots$ F.A.E., $\mathbb{E}(X_i) = \mu$, $Var(X_i) = \sigma^2$,

$S_n := X_1 + \ldots + X_n$, AKKOR $\forall x \in \mathbb{R}$:

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{S_n - n \cdot \mu}{\sigma \cdot \sqrt{n}} < x\right) = \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy$$

(SPEC ESET: HA $\mathbb{P}(X_i = 1) = p$, $\mathbb{P}(X_i = 0) = 1 - p$, AKKOR VISSZAKAPJUK A DE MOIVRE-LAPLACE-TÉTELT)

ALKALMAZÁS: $X_i \sim POI(1)$, AKKOR $\boxed{S_n \sim POI(n)}$

ÍGY $\mathbb{P}\left(\frac{S_n - n}{\sqrt{n}} < x\right) \overset{\circledast}{\underset{\text{C.N.T.}}{\approx}} \Phi(x)$, HA n NAGY.

(HISZEN $\mathbb{E}(X_i) = 1$, $Var(X_i) = 1$)

$\boxed{12.2}$ $X_i \sim EXP\left(\frac{1}{5}\right)$ $\boxed{\mu = 5}$ $\boxed{\sigma^2 = 25}$ $\boxed{CHT}$

$$\mathbb{P}(S_{100} > 525) = \mathbb{P}\left(\frac{S_{100} - 5\cdot 100}{\sqrt{25}\cdot\sqrt{100}} > \frac{525 - 5\cdot 100}{\sqrt{25}\cdot\sqrt{100}}\right) \approx$$

$$\approx \mathbb{P}\left(Y > \frac{25}{50}\right) = 1 - \Phi\left(\frac{1}{2}\right) = 1 - 0.6915 = 0.3085$$

ANOL $Y \sim N(0,1)$

MÁSIK MEGOLDÁS: HA ∞ SOK ÉGŐNK LENNE, AKKOR A CSERE-IDŐPONTOK EGY $1/5$ INTENZITÁSÚ POISSON FOLYAMATOT ALKOTNÁNAK.
ÍGY AZ 525 ÓRA ALATT ELHASZNÁLT ÉGŐK SZÁMA $N \sim POI(105)$ ELOSZLÁSÚ.

$$\mathbb{P}(N < 100) = \mathbb{P}\left(\frac{N - 105}{\sqrt{105}} < \frac{100 - 105}{\sqrt{105}}\right) \overset{\circledast}{\approx}$$

$$\approx \Phi\left(\frac{-5}{\sqrt{105}}\right) = \Phi(-0.488) = 1 - \Phi(0.488) \approx$$

$$1 - 0.6879 = 0.3121$$

$\boxed{12.3}$ $\mathbb{E}(X_i) = 74$, $\sqrt{Var(X_i)} = 14$

a) $\mathbb{P}\left(\frac{S_{25}}{25} \geq 80\right) = \mathbb{P}(S_{25} \geq 2000) =$

$$\mathbb{P}\left(\frac{S_{25} - 25\cdot 74}{\sqrt{25}\cdot 14} \geq \frac{2000 - 25\cdot 74}{\sqrt{25}\cdot 14}\right) \overset{C.H.T}{\approx} \mathbb{P}\left(Y \geq \frac{150}{70}\right)$$

$$= 1 - \Phi\left(\frac{15}{7}\right) = 1 - 0.9838 = 0.0162$$

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b) $\frac{1}{2}$, ÉS EZ A c) MEGOLDÁSÁBÓL FOG LÁTSZANI:

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c) $S_{25} = X_1 + \ldots + X_{25}$, $S^*_{64} = X^*_1 + \ldots + X^*_{64}$

$$Y := \frac{S_{25} - 25\cdot 74}{5\cdot 14} \qquad Y^* := \frac{S^*_{64} - 64\cdot 74}{8\cdot 14}$$

$$\mathbb{P}\left(\frac{S_{25}}{25} + 2.2 < \frac{S^*_{64}}{64}\right) =$$

$$\mathbb{P}\left(\frac{5\cdot 14\cdot Y + 25\cdot 74}{25} + 2.2 < \frac{8\cdot 14\cdot Y^* + 64\cdot 74}{64}\right) =$$

$$\mathbb{P}\left(\frac{14}{5}\cdot Y + 2.2 < \frac{14}{8}\cdot Y^*\right) = \mathbb{P}\left(2.2 < \frac{14}{8}\cdot Y^* - \frac{14}{5}\cdot Y\right) = ☺$$

C.H.T. $\Rightarrow Y \approx N(0,1)$, $Y^* \approx N(0,1)$, F.A.E.

ÍGY $\underbrace{\left(\frac{14}{8}\cdot Y^* - \frac{14}{5}\cdot Y\right)}_{Z} \sim N\left(0, \left(\frac{14}{8}\right)^2 + \left(\frac{14}{5}\right)^2\right) = N(0, \sigma^2)$

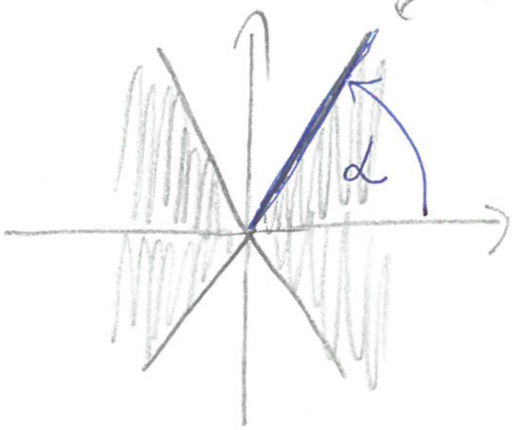
$\boxed{\sigma = 3.3}$ ☺ $= \mathbb{P}(2.2 < Z) = 1 - \Phi\left(\frac{2.2}{3.3}\right)$

$\boxed{12.3}$ (d) $\mathbb{P}\left(\left|\frac{S_{64}^*}{64}-74\right| < \left|\frac{S_{25}}{25}-74\right|\right) =$

$= \mathbb{P}\left(\left|\frac{14}{8}\cdot Y^*\right| < \left|\frac{14}{5}\cdot Y\right|\right) = \mathbb{P}\left(|Y^*| < \frac{8}{5}\cdot |Y|\right) = ✪$

(Y, Y^*) KÉTDIMENZIÓS STANDARD NORMÁLIS

KÉK EGYENES MEREDEKSÉGE: $\frac{8}{5}$



$\mathbb{P}\left((Y, Y^*) \in \text{SATÍROZOTT}\right)$

$\alpha = \arctan\left(\frac{8}{5}\right) = 1.01$

(Y, Y^*) SZÖGE EGYENLETES ELOSZLÁSÚ

A $[0, 2\pi]$ INTERVALLUMON (LÁSD: $\boxed{11.4}$ (b))

$✪ \overset{\text{C.H.T.}}{\approx} \frac{4\alpha}{2\pi} = 0.64$

$\boxed{12.4}$ $\mathbb{E}(X_i) = \mu$, $\sqrt{Var(X_i)} = 2$

$0.95 \overset{\text{KELL}}{=\!=} \mathbb{P}\left(\left|\frac{S_n}{n}-\mu\right| \le 0.5\right) = \mathbb{P}\left(\left|\frac{S_n - n\cdot\mu}{2\cdot\sqrt{n}}\right| \le \frac{0.5\cdot\sqrt{n}}{2}\right)$

$\overset{\text{C.H.T.}}{\approx} \mathbb{P}\left(|Y| \le \frac{\sqrt{n}}{4}\right) = \Phi\left(\frac{\sqrt{n}}{4}\right) - \Phi\left(-\frac{\sqrt{n}}{4}\right) =$

$= 2\cdot\Phi\left(\frac{\sqrt{n}}{4}\right) - 1$, TEHÁT KELL: $0.975 = \Phi\left(\frac{\sqrt{n}}{4}\right)$

AHOL $Y \sim N(0,1)$

$\frac{\sqrt{n}}{4} = \Phi^{-1}(0.975) = 1.96$, ÍGY: $\boxed{n = 62}$

$\boxed{12.5}$ $X \sim BIN(60, \frac{1}{2})$ $\mathbb{E}(X) = 30$, $Var(X) = 15$

a)

$$\mathbb{P}(|X - 30| \geq 20) \overset{\text{CSEB.}}{\leq} \frac{Var(X)}{(20)^2} = 0.0375$$

DE MOIVRE–LAPLACE/C.H.T. NEM ALKALMAZNATÓ,

MERT $\mathbb{P}\left(\left| \frac{X-30}{\sqrt{15}} \right| \geq \frac{20}{\sqrt{15}} \right) = \mathbb{P}(|Y| \geq 5.16) \approx$

$\boxed{\text{ILYET NEM SZABAD}}$

$$\approx 2 \cdot (1 - \underbrace{\Phi(5.16)}_{\text{RAJTA SINCS A TÁBLÁZATUNKON.}}) \approx 2 \cdot (1-1) = 0$$

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b) LEGYEN $X_i = \mathbb{1}[i\text{-EDIK DOBÁS FEJ}]$

$X = X_1 + \ldots + X_{60}$ $\mathbb{E}(Y_\beta) = \mathbb{E}(e^{\beta \cdot X}) =$

$\mathbb{E}(e^{\beta \cdot X_1} \cdot e^{\beta \cdot X_2} \cdot \ldots \cdot e^{\beta \cdot X_{60}}) =$ FÜGGETLENSÉG

$\mathbb{E}(e^{\beta \cdot X_1}) \cdot \mathbb{E}(e^{\beta \cdot X_2}) \cdot \ldots + \mathbb{E}(e^{\beta \cdot X_{60}}) =$

$\left(\frac{1}{2} \cdot e^{\beta \cdot 0} + \frac{1}{2} \cdot e^{\beta \cdot 1} \right)^{60} = 2^{-60} \cdot (1 + e^{\beta})^{60} = M_X(\beta)$

(X MOMENTUMGENERÁLÓ F.V.-E A β HELYEN)

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c) $\mathbb{P}(X \geq 50) \overset{\boxed{\beta \geq 0}}{=} \mathbb{P}(e^{\beta \cdot X} \geq e^{\beta \cdot 50}) = \mathbb{P}(Y_\beta \geq e^{\beta \cdot 50})$

$$\overset{\text{MARKOV}}{\leq} \frac{\mathbb{E}(Y_\beta)}{e^{\beta \cdot 50}} = \frac{2^{-60} \cdot (1+e^{\beta})^{60}}{e^{\beta \cdot 50}}$$

d) $$\min_{\beta>0}\left\{\frac{2^{-60}\cdot(1+e^{\beta})^{60}}{e^{\beta\cdot 50}}\right\} = \min_{\beta>0} g(\beta) = ?$$

$$f(\beta) := \frac{1}{60}\cdot \ln\big(g(\beta)\big) =$$

$$= -\ln(2) + \ln(1+e^{\beta}) - \frac{5}{6}\beta$$

$$f'(\beta) = \frac{e^{\beta}}{1+e^{\beta}} - \frac{5}{6} \qquad f''(\beta) = \frac{e^{\beta}}{(1+e^{\beta})^2} > 0$$

(TEHÁT f KONVEX FÜGGVÉNY)

$$f'(\beta) = 0 \iff \frac{e^{\beta}}{1+e^{\beta}} = \frac{5}{6} \iff e^{\beta^*} = 5 \iff \boxed{\beta^* = \ln(5)}$$

$$g(\beta^*) = \frac{2^{-60}\cdot(1+5)^{60}}{5^{50}} = \frac{3^{60}}{5^{50}}$$

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e) $$\mathbb{P}(|X-30| \geq 20) = \underbrace{\mathbb{P}(X \geq 50)}_{\leq \frac{3^{60}}{5^{50}}} + \underbrace{\mathbb{P}(X \leq 10)}_{☹}$$

☹ $= \mathbb{P}($ÍRÁSOK SZÁMA $\geq 50)$

$= \mathbb{P}($ FEJEK —"— $\geq 50) \leq 3^{60}/5^{50}$, TEHÁT

$$\mathbb{P}(|X-30| \geq 20) \leq 2\cdot\frac{3^{60}}{5^{50}} < 10^{-6}$$ (SOKKAL JOBB, MINT AMIT AZ a) RÉSZBEN KAPTUNK)

$\boxed{12.6}$ AZ a) RÉSZT MÁR MEGOLDOTTUK: $\boxed{9.3}$

$i = 1, 2, \ldots, 220$

$A_i := \{$ i-EDIK NAP ODAFELÉ LEKÉSI $\}$

$B_i := \{$ —''— —''— HAZAFELÉ —''— $\}$

$\boxed{9.3}_{a)} \Rightarrow \mathbb{P}(A_i) = 0.1587$

HASONLÓAN: $\mathbb{P}(B_i) = 0.0808$

LEGYEN $X_i := \mathbb{1}[A_i]$, $Y_i := \mathbb{1}[B_i]$,

$Z_i := X_i - Y_i$, $S_{220} := Z_1 + \ldots + Z_{220}$

$\mathbb{P}($ HAZAFELÉ TÖBBSZÖR KÉSI LE, MINT ODAFELÉ $) =$

$\mathbb{P}(S_{220} < 0) = ?$

$\mu = \mathbb{E}(Z_i) = \mathbb{E}(X_i) - \mathbb{E}(Y_i) = \mathbb{P}(A_i) - \mathbb{P}(B_i) = 0.0779$

$\sigma^2 = Var(Z_i) \underset{\substack{\uparrow \\ \text{FÜGGETLENSÉG}}}{=} Var(X_i) + Var(Y_i) = \mathbb{P}(A_i)\cdot(1-\mathbb{P}(A_i)) + \mathbb{P}(B_i)\cdot(1-\mathbb{P}(B_i)) = 0.2077$

TEHÁT $\sigma = 0.456$

$$? = \mathbb{P}\left(\frac{S_{220} - 220\cdot\mu}{\sqrt{220}\cdot\sigma} < \frac{-220\cdot\mu}{\sqrt{220}\cdot\sigma} \right) \overset{\text{C.H.T.}}{\approx}$$

$$\mathbb{P}\left(\underset{\substack{\uparrow \\ N(0,1)}}{Y} < \frac{-220\cdot\mu}{\sqrt{220}\cdot\sigma} \right) = 1 - \Phi(2.6) = 1 - 0.9953 = 0.0047$$