

NAME: ..... NEPTUN-CODE: .....

**Probability theory 1 exam, 2023. Dec. 19.**

*Working time: 100 min. Only simple, non-programmable calculators are allowed, standard normal distribution table on the other side.*

*The achievable maximum score (with the Bonus exercise) is 110 points, but we consider 100 points as 100%.*

- T. 1.** (a) (2+4 points) Define the notion of variance and prove the formula about the value of  $\text{Var}(aX + b)$  (where  $a, b \in \mathbb{R}$  and  $X$  is a random variable).
- (b) (2+4 points) State a necessary and sufficient condition for  $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$  to hold using the notion of covariance! Prove this statement.
- (c) (3+3 points) Calculate the expected value and the variance of a random variable  $X$  with binomial distribution by writing  $X$  as a sum of indicator variables.
- T. 2.** Let  $X$  denote a random variable with  $\text{EXP}(\lambda)$  distribution.
- (a) (2+6 points) Write down the probability density function of  $X$  and calculate the moment generating function of  $X$ .
- (b) (5 points) Let  $\psi(x) = e^{-\lambda x}$ . What is the distribution of  $\psi(X)$ ?
- (c) (3+2 points) State and prove the memoryless property of the  $\text{EXP}(\lambda)$  distribution.
- T. 3.** (a) (4 points) State and prove Markov's inequality!
- (b) (5 points) State and prove Chebyshev's inequality!
- (c) (5 points) State and prove the weak law of large numbers!

*Formula sheet:  $X \sim \text{EXP}(\lambda): \mathbb{E}(X) = \frac{1}{\lambda}, \text{Var}(X) = \frac{1}{\lambda^2}$ .  $X \sim \text{GEO}(p): \mathbb{E}(X) = \frac{1}{p}, \text{Var}(X) = \frac{1-p}{p^2}$ .  $X \sim \text{POI}(\lambda): \mathbb{E}(X) = \text{Var}(X) = \lambda$ .*

- P. 1.** In a town two taxi companies operate, one uses yellow cabs, the other white cabs. 10% of the taxis are white, the rest of them are yellow. A traffic accident caused by a taxi driver was witnessed by a bystander, who claims that the culprit was driving a white taxi. Subsequent police investigation found that the reliability of the witness is 80%, meaning that under similar circumstances the witness can correctly identify the colour of a taxi in 80% of the cases.
- (a) (8 points) Given the above info, what is the probability that the culprit was indeed driving a white taxi?
- (b) (8 points) Another independent witness (also with 80% reliability) shows up and confirms the claim of the first witness. Now what is the probability that the accident was caused by a white taxi?
- P. 2.** (16 points) The sides of a rectangle have length 1 and 2. On both of the sides of length 1 we pick one point each, independently, with uniform distribution. Let  $Z$  denote the distance of these two points. Determine the cumulative distribution function of  $Z$ .
- P. 3.** An ice cream stand sells two types of ice cream: chocolate and vanilla. The customers arrive according to a Poisson point process, where the expected time between two consecutive customers is two minutes. Each customer chooses between the two types of ice cream with a fair coin toss.
- (a) (7 points) Identify the cumulative distribution function of the arrival time of the first customer who buys chocolate after the ice cream stand opens.
- (b) (11 points) Use the central limit theorem to estimate the probability that the number of customers that arrive between 8am and 2pm differs from the number of customers that arrive between 2pm and 8pm by at most ten.

**Bonus:** (10 point) Estimate the probability that between 8am and 2pm more people buy vanilla than chocolate, but between 8am and 8pm more people buy chocolate than vanilla.

