NAME: .... NEPTUN-CODE: ....

Probability theory 1 exam, 2024. Jan. 23.
Working time: 100 min. Only simple, non-programmable calculators are allowed,
standard normal distribution table on the other side.
The achievable maximum score (with the Bonus exercise) is 110 points, but we consider 100 points as 100%.

T. 1. (a) *(2+2+3+3 points)* Write down the probability weight function of the *binomial* and *Poisson* distributions and calculate (from the weight function or using another learned method) the expected value of both distributions.

 (b) *(7 points)* In what limiting case and exactly in what sense can the binomial distribution be approximated by the Poisson distribution? State and prove the theorem you have learned about this.

T. 2. (a) *(6 points)* State and prove the Cauchy-Schwarz inequality for the random variables X and Y.

 (b) *(6 points)* How do we define the correlation coefficient $\rho(X, Y)$ of X and Y? What is the maximum possible value of it? In what case do we get this extreme value for $\rho(X, Y)$? Prove your claims.

 (c) *(6 points)* State and prove Steiner's theorem!

T. 3. Let X and Y be independent random variables with probability density functions $f_X(x)$ and $f_Y(y)$, and let $Z = X + Y$.

 (a) *(8 points)* What formula gives the cumulative distribution function and probability density function of Z? Prove your claims.

 (b) *(2 points)* How do we define the moment generating function of X?

 (c) *(5 points)* How do we get the moment generating function of Z from the moment generating functions of X and Y? Prove your claim.

P. 1. 60% of the border guards of Ruritania are superficial, while 40% of them are thorough. Superficial border guards stop cars that try to enter the county with probability 0.6, thorough ones stop cars with probability 0.9, independently of each other. I want to drive into Ruritania, and I see a blue car and a red car in front of me. The border guard stopped the blue car, but did not stop the red car.

 (a) *(9 points)* What is the probability that this border guard is superficial in nature?

 (b) *(8 points)* Knowing the fate of the previous two cars, what is the probability that this border guard will stop me?

P. 2. *(16 points)* The lifetime of the „Mehr Licht" type light bulb is exponentially distributed. According to the manufacturer's measurements, 80 percent of the light bulbs survive for at least one year. How should the manufacturer set the expiration date (i.e., the time limit of guaranteed performance) of the light bulb if they want no more than 2 percent of customers to complain?

 Bonus: *(10 points)* I use this type of light bulb in my study and replace it immediately when it burns out. Determine the distribution function for the time of the fourth light bulb change (if year is the unit of measurement of time)! *Hint*: What do the lightbulb change times form?

P. 3. Let $X_1, X_2, \ldots, X_n, X_{n+1}$ be independent random variables with $\mathcal{N}(0, 1)$ distribution, and define the random variable Y with the formula $Y = \frac{X_1 + \cdots + X_n}{n}$.

 (a) *(6 points)* Determine the covariance matrix of the random two-dimensional vector (X_1, Y).

 (b) *(11 points)* $\mathbb{P}(|Y| \leq |X_{n+1}|) =$?

$$\Phi(z)$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000