

NAME: NEPTUN-CODE:

Probability theory 1 exam, 2024. Jan. 9.

Working time: 100 min. Only simple, non-programmable calculators are allowed, standard normal distribution table on the other side.

The achievable maximum score (with the Bonus exercise) is 110 points, but we consider 100 points as 100%.

- T. 1.** (a) (5 points) Define when the events A_1, \dots, A_n are mutually independent.
- (b) (6 points) Construct events A, B, C such that $\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C) = 1/2$, moreover A, B, C are pairwise independent, but not mutually independent.
- (c) (5 points) Let us assume $\mathbb{P}(B) > 0$. Show that A and B are independent if and only if $\mathbb{P}(A) = \mathbb{P}(A|B)$.
- T. 2.** Assume that the random variable X is absolutely continuous, let us denote its cumulative distribution function (c.d.f.) by F and its probability density function (p.d.f.) by f .
- (a) (8 points) Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable, strictly monotone bijection. Let $Y = \psi(X)$. Write down and prove the formula that expresses the c.d.f. G and the p.d.f. g of Y in terms of F, f and ψ .
- (b) (9 points) Let $X \sim \mathcal{N}(0, 1)$. Let $Y = X^4$. Calculate the p.d.f. of Y .
- T. 3.** (a) (3 points) Define the moment generating function $t \mapsto M(t)$ of the random variable X .
- (b) (2+5 points) Write down the formula that connects the first/second derivative of M and the first/second moment of X . Prove the formula relating the the first derivative of M and the expectation of X in the case when X is absolutely continuous and f denotes its p.d.f.
- (c) (7 points) Derive the formula for the moment generating function M of the standard normal distribution and calculate the variance of the standard normal distribution using M .

Formula sheet: $X \sim \text{EXP}(\lambda): \mathbb{E}(X) = \frac{1}{\lambda}, \text{Var}(X) = \frac{1}{\lambda^2}$. $X \sim \text{POI}(\lambda): \mathbb{E}(X) = \text{Var}(X) = \lambda$.

- P. 1.** (17 points) The numbers $1, 2, \dots, 6$ are distributed randomly (without repetition) among six players, A, B, C, D, E and F. First A and B compete: whoever has the higher number wins the round. The one that wins the first round will now face C in the second round, then the one that wins this round will play against D, then the winner of this round competes with E, then with the winner of this round competes with F. Let X denote the number of rounds that A wins. Find $\mathbb{E}(X)$.
- P. 2.** (16 points) A factory produces two types of coins: a fair one and a rigged one, which has a probability of 56% showing heads. We have such a coin, but we don't know if it is fair or rigged. To decide about this, we perform the following statistical test: we flip the coin 1000 times, and if it shows heads at least 530 times, we declare it to be rigged, but if there are less than 530 heads among the coin tosses then we declare the coin to be fair. What is the probability that our test is wrong if the coin is rigged?
- P. 3.** (8+9 points) In the city of Quahog, the fire department receives phone calls for two possible reasons: false fire alarms by pranksters and actual fires. These phone calls arrive according to independent Poisson point processes. On average, there are two actual fires per week. On average, 12 hours pass between two consecutive prank phone calls. Let X denote the number of prank calls before the first actual fire, starting from now. Compute $\mathbb{E}(X)$ and $\text{Var}(X)$. *Hint: First compute $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$ using the tower rule of conditional expectation by conditioning on the time of the first real fire.*

Bonus: (10 points) What is the distribution of X ?

