VALSZÁM 1. VIZSGA, 2024. JAN. 16.

$\boxed{E1}$ a) HA $A_1, A_2, \dots, A_m$ TELJES ESEMÉNYRENDSZER:

$$\mathbb{P}(B) = \sum_{k=1}^{m} \mathbb{P}(B \cap A_k) = \sum_{k=1}^{m} \mathbb{P}(B|A_k) \cdot \mathbb{P}(A_k)$$

b) $$\mathbb{P}(A_k|B) = \frac{\mathbb{P}(A_k \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B|A_k) \cdot \mathbb{P}(A_k)}{\sum_{l=1}^{m} \mathbb{P}(B|A_l) \cdot \mathbb{P}(A_l)}$$

c) $$\mathbb{P}(E_4) = \mathbb{P}(E_4 \cap E_3) = \mathbb{P}(E_4|E_3) \cdot \mathbb{P}(E_3)$$
$$\mathbb{P}(E_3) = \mathbb{P}(E_3 \cap E_2) = \mathbb{P}(E_3|E_2) \cdot \mathbb{P}(E_2)$$
$$\mathbb{P}(E_2) = \mathbb{P}(E_2 \cap E_1) = \mathbb{P}(E_2|E_1) \cdot \mathbb{P}(E_1)$$

TEHÁT: $\mathbb{P}(E_4) = \mathbb{P}(E_4|E_3) \cdot \mathbb{P}(E_3|E_2) \cdot \mathbb{P}(E_2|E_1) \cdot \mathbb{P}(E_1)$

$\boxed{E2}$ a) FÜGGETLEN KÍSÉRLETEK

EGY KÍSÉRLET p VALSÉGGEL SIKER

X = PRÓBÁLKOZÁSOK SZÁMA AZ ELSŐ SIKERIG

$X \sim GEO(p)$ $\quad \mathbb{P}(X = k) = \underbrace{(1-p)^{k-1}}_{\substack{k-1 \\ \text{SIKERTELEN}}} \cdot p$ $\quad$ k-ADIK SIKERES

$k = 1, 2, \dots$

$$M(t) = \mathbb{E}(e^{t \cdot X}) = \sum_{k=1}^{\infty} (1-p)^{k-1} \cdot p \cdot e^{t \cdot k} =$$

$$= p\cdot e^t \cdot \sum_{k=1}^{\infty} \left((1-p)\cdot e^t\right)^{k-1} = p\cdot e^t \cdot \sum_{\ell=0}^{\infty} \left((1-p)\cdot e^t\right)^{\ell}$$

$$= \begin{cases} +\infty & \text{HA } \boxed{(1-p)\cdot e^t \geq 1} \\ (p\cdot e^t)/(1-(1-p)\cdot e^t) = \dfrac{p}{e^{-t}-(1-p)} & \text{HA } \boxed{(1-p)\cdot e^t < 1} \end{cases}$$

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b) FÜGGETLEN KÍSÉRLETEK
EGY KÍSÉRLET p VALÓSZÍNŰSÉGGEL SIKER

Y = PRÓBÁLKOZÁSOK SZÁMA AZ r-EDIK SIKERIG

$Y \sim NEGBIN(r, p)$

$$\mathbb{P}(Y=k) = \underbrace{\binom{k-1}{r-1}}_{\boxed{\text{ELSŐ } r-1 \text{ SIKER LEHETSÉGES POZÍCIÓI}}} \cdot \underbrace{(1-p)^{k-r}\cdot p^{r}}_{\text{ELSŐ } k \text{ PRÓBÁLKOZÁSBÓL } r \text{ SIKER, } k-r \text{ KUDARC}}, \quad k = r, r+1, \ldots$$

$Y = X_1 + \ldots + X_r$, AHOL $X_1, \ldots, X_r$ F.A.E. $GEO(p)$

$$M_Y(t) = \mathbb{E}(e^{t\cdot Y}) = \mathbb{E}\left(e^{t\cdot(X_1+\ldots+X_r)}\right) \overset{\text{FÜGGETLENSÉG}}{=}$$

$$\mathbb{E}(e^{t\cdot X_1})\cdot\mathbb{E}(e^{t\cdot X_2})\cdot\ldots\cdot\mathbb{E}(e^{t\cdot X_r}) =$$

$$\mathbb{E}(e^{t\cdot X})^r = \left(\frac{p}{e^{-t}-(1-p)}\right)^r$$

$\boxed{\text{ELM 3}}$ a) $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}, \quad E(X|Y=y) = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\,dx$$

b) $f_X(x) = \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{(x-\mu)^2}{2}\right)$

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{(y-x)^2}{2}\right)$$

$$f(x,y) = f_X(x) \cdot f_{Y|X}(y|x) = \frac{1}{2\pi} \cdot \exp\left(-\frac{(x-\mu)^2+(y-x)^2}{2}\right)$$

$$\boxed{Y \sim N(\mu, 2)} \Rightarrow f_Y(y) = \frac{1}{\sqrt{2}\sqrt{2\pi}} \cdot \exp\left(-\frac{(y-\mu)^2}{2\cdot 2}\right)$$

$$f_{X|Y}(x|y) = \frac{\frac{1}{2\pi} \cdot \exp\left(-\frac{(x-\mu)^2+(y-x)^2}{2}\right)}{\frac{1}{\sqrt{2}\sqrt{2\pi}} \cdot \exp\left(-\frac{(y-\mu)^2}{4}\right)} =$$

$$= \frac{\sqrt{2}}{\sqrt{2\pi}} \cdot \exp\left(\frac{(y-\mu)^2 - 2\cdot(x-\mu)^2 - 2\cdot(y-x)^2}{4}\right) =$$

$$= \frac{1}{\frac{1}{\sqrt{2}} \cdot \sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot \frac{\left(x - \left(\frac{y+\mu}{2}\right)\right)^2}{1/2}\right)$$

TEHÁT $Y = y$ FELTÉTEL MELLETT X FELTÉTELES ELOSZLÁSA $N\left(\frac{y+\mu}{2}, \frac{1}{2}\right)$

3. OLDAL

EMIATT: $\mathbb{E}(X \mid Y = y) = \frac{1}{2}\cdot(y+\mu)$

$\boxed{\text{GYAK 1}}$ a) ÉS b) ESETBEN IS: $\boxed{p := I}$

$X_n \sim BIN(n, p)$ $\boxed{\varepsilon := 0.05}$

$$\mathbb{P}\left(\left|\frac{X_n}{n} - p\right| > \varepsilon\right) = \mathbb{P}\left(\left|\frac{X_n - n\cdot p}{n}\right| > \varepsilon\right) =$$

$$\mathbb{P}\left(\left|\frac{X_n - n\cdot p}{\sqrt{n}\cdot\sqrt{p\cdot(1-p)}}\right| > \frac{\sqrt{n}\cdot\varepsilon}{\sqrt{p\cdot(1-p)}}\right) \overset{\text{DE MOIVRE}}{\approx} \mathbb{P}\left(|X^*| > \frac{\sqrt{n}\cdot\varepsilon}{\sqrt{p\cdot(1-p)}}\right)$$

$$= 1 - \left(2\cdot\Phi\left(\frac{\sqrt{n}\cdot\varepsilon}{\sqrt{p\cdot(1-p)}}\right) - 1\right) = \underbrace{2\cdot\left(1 - \Phi\left(\frac{\sqrt{n}\cdot\varepsilon}{\sqrt{p\cdot(1-p)}}\right)\right)}$$

KELL: OLYAN n, HOGY $0.01 \leq$ — " —

KELL: $\Phi\left(\frac{\sqrt{n}\cdot\varepsilon}{\sqrt{p\cdot(1-p)}}\right) \geq 0.995$ TÁBLÁZAT:

KELL: $\frac{\sqrt{n}\cdot\varepsilon}{\sqrt{p\cdot(1-p)}} \geq 2.58$

KELL: $\boxed{n \geq \frac{(2.58)^2\cdot p\cdot(1-p)}{\varepsilon^2}}$

a) $I = p = \frac{1}{5}$

ÍGY

$n \geq 426.0096$

$\boxed{n \geq 427}$

b) LEGROSSZABB ESET: $\boxed{p = 1/2}$, ÍGY $n \geq 665.64$

$\boxed{n \geq 666}$

(4. OLDAL

$\boxed{\text{GYAK 2}}$ $A_r := \{ r\text{-ADIK ÉS } r+1\text{-EDIK ÉV BŐ} \}$

$B_r := \{ \text{—''— —''— —''— —''— SZŰK} \}$

$$X_m = \sum_{r=1}^{m} \mathbb{1}[B_r] \quad , \quad Y_m = \sum_{r=1}^{m} \mathbb{1}[A_r]$$

$$\mathrm{Cov}(X_m, Y_m) \overset{\boxed{\text{BIL}}}{=} \sum_{k,l=1}^{m} \underbrace{\mathrm{Cov}(\mathbb{1}[B_k], \mathbb{1}[A_l])}_{=: C_{k,l}} = \star$$

$$C_{k,l} = \mathbb{P}(B_k \cap A_l) - \mathbb{P}(B_k) \cdot \mathbb{P}(A_l)$$

$\boxed{k=l}$: $C_{k,l} = 0 - \frac{1}{4} \cdot \frac{1}{4} = -\frac{1}{16}$

$\boxed{|k-l|=1}$: $C_{k,l} = 0 - \frac{1}{4} \cdot \frac{1}{4} = -1/16$

$\boxed{|k-l| \geq 2}$: $\boxed{C_{k,l} = 0}$ ⟵ FÜGGETLENSÉG MIATT

$$\star = (m + 2 \cdot (m-1)) \cdot \left(-\frac{1}{16}\right) \approx -\frac{3}{16} \cdot m$$

$$\mathrm{Var}(X_m) = \mathrm{Cov}(X_m, X_m) = \sum_{k,l=1}^{m} \underbrace{\mathrm{Cov}(\mathbb{1}[B_k], \mathbb{1}[B_l])}_{=: \tilde{C}_{k,l}}$$

$$\tilde{C}_{k,l} = \mathbb{P}(B_k \cap B_l) - \mathbb{P}(B_k) \cdot \mathbb{P}(B_l)$$

$\boxed{k=l}$: $\tilde{C}_{k,l} = \mathbb{P}(B_k) - \mathbb{P}(B_k)^2 = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$

$\boxed{|k-l|=1}$: $\tilde{C}_{k,l} = \frac{1}{8} - \left(\frac{1}{4}\right)^2 = \frac{1}{16}$

$\boxed{|k-l| \geq 2}$: $\boxed{\tilde{C}_{k,l} = 0}$

$$\mathrm{Var}(X_n) = \sum_{k,l=1}^{n} \tilde{C}_{k,l} = n \cdot \frac{3}{16} + 2\cdot(n-1)\cdot \frac{1}{16} \approx \frac{5}{16} \cdot n$$

$\mathrm{Var}(X_n) = \mathrm{Var}(Y_n)$ SZIMMETRIA MIATT

$$\mathrm{Corr}(X_n, Y_n) = \frac{\mathrm{Cov}(X_n, Y_n)}{\sqrt{\mathrm{Var}(X_n)}\cdot\sqrt{\mathrm{Var}(Y_n)}} = \frac{\mathrm{Cov}(X_n, Y_n)}{\mathrm{Var}(X_n)} \approx$$

$$\approx \frac{-3/16 \cdot n}{5/16 \cdot n} = -\frac{3}{5}$$

TEHÁT $\boxed{\lim_{n\to\infty} \mathrm{Corr}(X_n, Y_n) = -\frac{3}{5}}$

$\boxed{\text{GYAK 3}}$ X ÉS Y FÜGGETLENEK

$\boxed{X^* := X - 1}$ $\boxed{Y^* := Y/2}$ $\boxed{\text{EKKOR } (X^*, Y^*) \text{ 2-DIM STANDARD NORMÁLIS}}$

a) $\mathbb{P}((X,Y) \in M_1) = \mathbb{P}(|X| \le 1, |Y| \ge 1) =$

$\mathbb{P}(|X^*+1| \le 1, |2Y^*| \ge 1) =$

$\mathbb{P}(-2 \le X^* \le 0) \cdot \mathbb{P}(Y^* \ge \frac{1}{2} \text{ VAGY } Y^* \le -\frac{1}{2})$

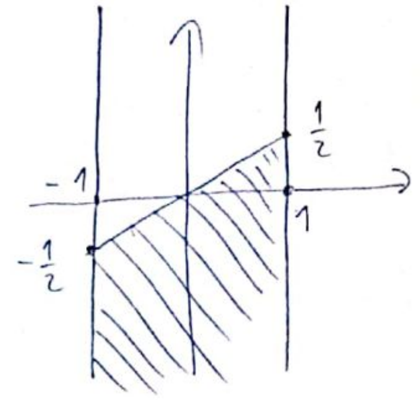
$= (\Phi(2) - \Phi(0)) \cdot 2 \cdot (1 - \Phi(\frac{1}{2}))$

$(0.9772 - \frac{1}{2}) \cdot 2 \cdot (1 - 0.6915) = 0.2944$

6. OLDAL

b) $\mathbb{P}(0 \le X' \le 2,\ Y' \le X' - 1) =$

$\mathbb{P}(-1 \le X^* \le 1,\ Y^* \le X^*/2) =$ $\boxed{\text{FORGÁS-SZIMM}}$



$= \frac{1}{2} \cdot \mathbb{P}(-1 \le X^* \le 1)$

$= \frac{1}{2} \cdot (2 \cdot \Phi(1) - 1) =$

$\Phi(1) - \frac{1}{2} = 0.3413$

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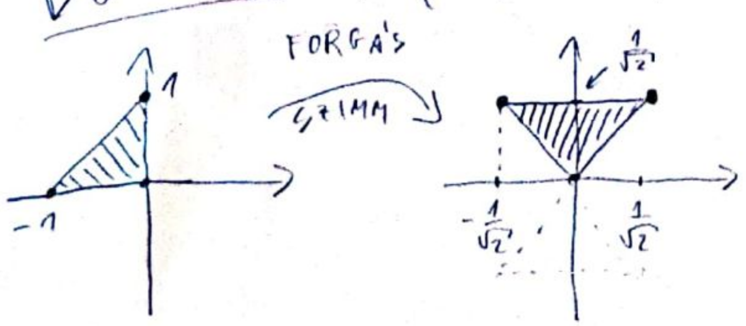
c) $\mathbb{P}\left(4 \cdot (X^* + 1)^2 - 8 \cdot (X^* + 1) + (2Y^*)^2 \le 0\right) =$

$\mathbb{P}\left((X^*)^2 + (Y^*)^2 \le 1\right) = \iint_{x^2+y^2 \le 1} \frac{1}{2\pi} \exp\left(-\frac{1}{2}(x^2+y^2)\right) dx\,dy$

$\underset{\text{POLÁR}}{=} \int_0^1 \int_0^{2\pi} \frac{1}{2\pi} \cdot \exp\left(-\frac{1}{2} r^2\right) d\varphi\, r\, dr =$

$\int_0^1 r \cdot \exp\left(-\frac{1}{2} r^2\right) dr = \left[-\exp\left(-\frac{1}{2} r^2\right)\right]_0^1 = 1 - e^{-1/2}$

BÓNUSZ: $\mathbb{P}(X^* \le 0,\ Y^* \ge 0,\ Y^* \le X^*) =$



$= \frac{1}{4} \cdot \mathbb{P}\left(-\frac{1}{\sqrt{2}} \le X^* \le \frac{1}{\sqrt{2}}\right)^2 =$

$= \frac{1}{4} \cdot \left(2 \cdot \Phi\left(\frac{1}{\sqrt{2}}\right) - 1\right)^2$

7. OLDAL

CamScannerrel szkennelve