VALÓSZÁM 1. VIZSGA, 2024. 01. 23.

$\boxed{E1}$ a) $X \sim BIN(n, p)$: $\mathbb{P}(X = k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$

$k = 0, 1, \dots, n$

$$M(t) = \mathbb{E}(e^{tX}) = \sum_{k=0}^{n} e^{t \cdot k} \cdot \mathbb{P}(X = k) =$$

$$\sum_{k=0}^{n} \binom{n}{k} \cdot (p \cdot e^t)^k \cdot (1-p)^{n-k} \xlongequal[\text{TÉTEL}]{\text{BINOMIÁLIS}} (p \cdot e^t + (1-p))^n$$

$$M'(t) = n \cdot (p \cdot e^t + (1-p))^{n-1} \cdot p \cdot e^t$$

$$\mathbb{E}(X) = M'(0) = n \cdot p$$

$Y \sim POI(\lambda)$: $\mathbb{P}(Y = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$, $k = 0, 1, 2, \dots$

$$\mathbb{E}(Y) = \sum_{k=0}^{\infty} k \cdot \mathbb{P}(Y = k) = \sum_{k=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{(k-1)!} =$$

$$= \lambda \cdot \sum_{k=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{k-1}}{(k-1)!} = \lambda \cdot \sum_{l=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^l}{l!} = \lambda$$

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b) TÉTEL: HA $X_n \sim BIN(n, p_n)$ ÉS $Y \sim POI(\lambda)$ ÉS $\lim_{n \to \infty} n \cdot p_n = \lambda$, AKKOR $\forall k \in \mathbb{N}_0$:

$$\boxed{\lim_{n \to \infty} \mathbb{P}(X_n = k) = \mathbb{P}(Y = k)}$$

BIZ: KÖV. OLDAL

1. OLDAL

BIZ: $\lim_{n\to\infty} \binom{n}{r} \cdot p_n^{r} \cdot (1-p_n)^{n-r} =$

$\lim_{n\to\infty} \frac{n\cdot(n-1)\cdot\ldots\cdot(n-r+1)}{r!} \cdot p_n^{r} \cdot (1-p_n)^{n} =$

$\lim_{n\to\infty} \frac{n^r}{r!} \cdot p_n^{r} \cdot (1-p_n)^{n} = \frac{1}{r!} \cdot \underbrace{\lim_{n\to\infty} (n\cdot p_n)^{r}}_{\lambda^r} \cdot \underbrace{\lim_{n\to\infty} (1-p_n)^{n}}_{e^{-\lambda}}$

$\boxed{E2}$ a) $|\mathbb{E}(X\cdot Y)| \leq \sqrt{\mathbb{E}(X^2)} \cdot \sqrt{\mathbb{E}(Y^2)}$

BIZ: $\underline{\underline{A}} = \begin{bmatrix} \mathbb{E}(X^2) & \mathbb{E}(X\cdot Y) \\ \mathbb{E}(X\cdot Y) & \mathbb{E}(Y^2) \end{bmatrix}$ ← EZ A MÁTRIX POZITÍV SZEMIDEFINIT:

$\underline{v}^T \cdot \underline{\underline{A}} \cdot \underline{v} = \mathbb{E}\left((v_1\cdot X + v_2 \cdot Y)^2\right) \geq 0$

TEHÁT $\boxed{\det(\underline{\underline{A}}) \geq 0}$

$\det(\underline{\underline{A}}) = \mathbb{E}(X^2)\cdot\mathbb{E}(Y^2) - (\mathbb{E}(X\cdot Y))^2 \geq 0$

ÁTRENDEZVE KAPJUK A CAUCHY-SCHWARZ-OT.

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b) $\varrho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\cdot\sqrt{Var(Y)}} \overset{a)}{\leq} 1$ ÉS

AKKOR ÉS CSAK AKKOR $= 1$, HA $Y = \alpha\cdot X + \beta$

VALAMILYEN $\alpha > 0$, $\beta \in \mathbb{R}$ ESETÉN

BIZ: HA $Y = \alpha \cdot X + \beta$ ÉS $\alpha > 0$, AKKOR

$\varrho(X,Y) = \frac{\alpha\cdot Var(X)}{\sqrt{Var(X)}\cdot\sqrt{\alpha^2\cdot Var(X)}} = \frac{\alpha}{|\alpha|} = 1$

(2. OLDAL

HA $\varrho(X,Y)=1$, AKKOR LEGYEN

$\sigma_X := \sqrt{Var(X)}$, $\sigma_Y := \sqrt{Var(Y)}$

$m_X := \mathbb{E}(X)$, $m_Y := \mathbb{E}(Y)$

$X^* := \dfrac{X - m_X}{\sigma_X}$, $Y^* := \dfrac{Y - m_Y}{\sigma_Y}$

$\boxed{Z_1 := X^* - Y^*}$ $Var(Z_1) = Var(X^*) + Var(Y^*) - 2\cdot Cov(X^*, Y^*)$

$= 1 + 1 - 2\cdot \varrho(X,Y) = 1+1-2 = 0$

TEHÁT $\mathbb{E}(Z_1) = 0$ MIATT $\mathbb{P}(Z_1 = 0) = 1$

ÍGY $X^* = Y^*$, ÍGY $Y = \dfrac{\sigma_Y}{\sigma_X}\cdot(X - m_X) + m_Y$

ÍGY $\boxed{\alpha = \dfrac{\sigma_Y}{\sigma_X}}$, $\boxed{\beta = m_Y - \dfrac{\sigma_Y}{\sigma_X}\cdot m_X}$

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c) $\forall a \in \mathbb{R}$: $\mathbb{E}((X-a)^2) = Var(X) + (m-a)^2$

AHOL $m = \mathbb{E}(X)$. BIZ: $\mathbb{E}((X-a)^2) =$

$\mathbb{E}((X - m + m - a)^2) = Var(X) + 2\cdot \underbrace{\mathbb{E}((X-m)\cdot(m-a))}_{0} +$

$+ (m-a)^2$ ✓

3. OLDAL

$\boxed{\text{ELM 3}}$ a) $F_{Z_1}(z) = \mathbb{P}(Z_1 \leq z) =$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbb{I}[x+y \leq z] \cdot f_X(x) \cdot f_Y(y)\, dy\, dx =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f_X(x) \cdot f_Y(y)\, dy\, dx$$

$$f_{Z_1}(z) = \frac{d}{dz} F_{Z_1}(z) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\, dx$$

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b) $M_X(t) = \mathbb{E}(e^{t \cdot X})$, $t \in \mathbb{R}$

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c) $M_{Z_1}(t) = \mathbb{E}(e^{t \cdot (X+Y)}) \overset{\text{L.O.T.U.S.}}{=}$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t \cdot (x+y)} \cdot f_X(x) \cdot f_Y(y)\, dy\, dx =$$

$$\left(\int_{-\infty}^{\infty} e^{t \cdot x} \cdot f_X(x)\, dx\right) \cdot \left(\int_{-\infty}^{\infty} e^{t \cdot y} \cdot f_Y(y)\, dy\right) =$$

$$= M_Y(t) \cdot M_X(t)$$

GYAK 1

a) $\mathbb{P}$(FELÜLETES | KÉKET MEG, PIROSAT NEM) =

$$= \frac{\mathbb{P}(\text{FELÜLETES ÉS KÉKET MEG, PIROSAT NEM})}{\mathbb{P}(\text{FELÜ, KÉKET MEG, PIR. NEM}) + \mathbb{P}(\text{ALAPOS, KÉKET +, PIROSAT } \emptyset)}$$

$$= \frac{(0.6)\cdot(0.6)\cdot(0.4)}{(0.6)\cdot(0.6)\cdot(0.4) + (0.4)\cdot(0.9)\cdot(0.1)} = \frac{0.144}{0.144 + 0.036}$$

$$= \frac{0.144}{0.18} = 0.8$$

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b) $\mathbb{P}$(MEGÁLLÍT | AMIT LÁTTAM) =

$$(0.6)\cdot(0.8) + (0.9)\cdot(1-0.8) = 0.66$$

GYAK 2 MÉRTÉKEGYSÉG: ÉV

T = KÖRTE ÉLETTARTAMA $\boxed{T \sim \mathrm{EXP}(\lambda)}$

$$0.8 = \mathbb{P}(T \geq 1) = e^{-1\cdot\lambda} = e^{-\lambda} \qquad \boxed{\lambda = -\ln(0.8)}$$

$\lambda \approx 0.22$

GARANCIA IDŐTARTAMA: $t \in \mathbb{R}_+$

$$0.02 = \mathbb{P}(T \leq t) = 1 - e^{-\lambda\cdot t}, \quad e^{-\lambda\cdot t} = 0.98$$

$$t = -\frac{\ln(0.98)}{\lambda} = \frac{\ln(0.98)}{\ln(0.8)} = 0.09053$$

ENNYI ÉV LEGYEN A GARANCIA

(5. OLDAL

BÓNUSZ: A KÖRTE-CSERE IDŐPONTOK EGY λ INTENZITÁSÚ POISSON PONT FOLYAMATOT ALKOTNAK.

$N_t :=$ A $[0,t]$ INTERVALLUMBA ESŐ KÖRTE-CSERÉK SZÁMA

$N_t \sim POI(t \cdot \lambda)$

$\boxed{T_4 := \text{NEGYEDIK KÖRTE-CSERE IDŐPONT}}$

$$F(t) = \mathbb{P}(T_4 \leq t) = \mathbb{P}(N_t \geq 4) =$$

$$1 - \mathbb{P}(N_t \leq 3) = 1 - \sum_{k=0}^{3} e^{-\lambda \cdot t} \cdot \frac{(\lambda t)^k}{k!}$$

AHOL $\boxed{\lambda = -\ln(0.8)}$

$\boxed{\text{GYAK 3}}$ a)

$Var(X_1)$	$Cov(X_1, Y)$
$Cov(X_1, Y)$	$Var(Y)$

$Var(X_1) = 1$

$\boxed{\text{FÜGGETLENSÉG}}$

$$Var(Y) \overset{\downarrow}{=} \frac{1}{n^2} \cdot (Var(X_1) + \ldots + Var(X_n)) = \frac{n}{n^2} = \frac{1}{n}$$

$$Cov(X_1, Y) \overset{BIL.}{=\!=} \frac{1}{n} \sum_{k=1}^{n} Cov(X_1, X_k) =$$

$$\frac{1}{n} \cdot \Big(\underbrace{cov(X_1, X_1)}_{1} + \underbrace{cov(X_1, X_2)}_{0} + \ldots + \underbrace{cov(X_1, X_n)}_{0} \Big) = \frac{1}{n}$$

6. OLDAL

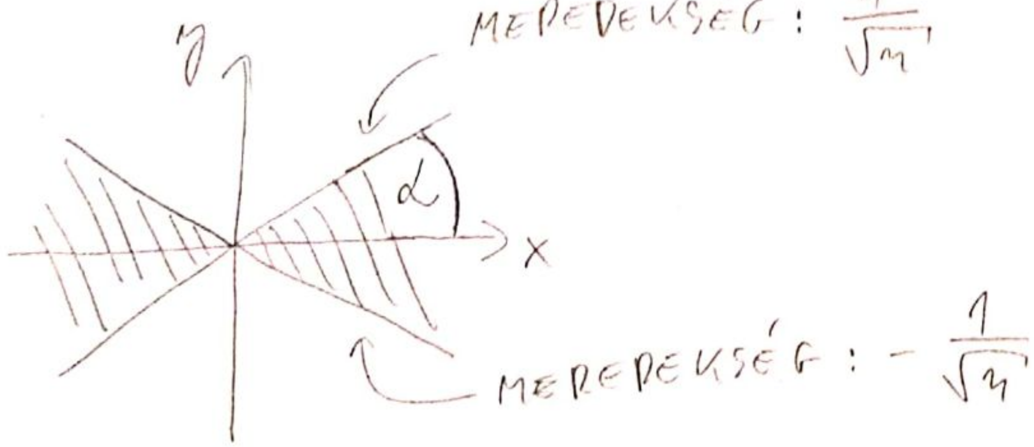
$\boxed{\text{GYAK 3}}$ b) $Y \sim N(0, \frac{1}{n})$, $X_{n+1} \sim N(0,1)$

$Y^* := \frac{1}{\sqrt{n}} \cdot Y$, $X'^* := X_{n+1}$

EKKOR (X'^*, Y^*) 2-DIM STANDARD NORMÁLIS

$\mathbb{P}(|Y| \leq |X_{n+1}|) = \mathbb{P}(\sqrt{n} \cdot |Y^*| \leq |X'^*|) =$

$\mathbb{P}(|Y^*| \leq \frac{1}{\sqrt{n}} \cdot |X'^*|) = \underbrace{\mathbb{P}((X'^*, Y^*) \in \text{SATÍROZOTT})}_{★}$



$$\tan(\alpha) = \frac{1}{\sqrt{n}} \Rightarrow \alpha = \arctan\left(\frac{1}{\sqrt{n}}\right)$$

2-DIM STANDARD NORMÁLIS SZÖGE EGYENLETES ELOSZLÁSÚ, ÍGY

$$★ = \frac{4 \cdot \alpha}{2\pi} = \frac{\arctan\left(\frac{1}{\sqrt{n}}\right)}{\pi/2}$$