Extreme value theory midterm exam, 15th Apr 2024

1. (a) What does it mean that a function is slowly varying at infinity?

   (b) Which of the functions

$$f(x) = (\log x)^2, \qquad g(x) = \frac{x^2}{\log x}, \qquad h(x) = \frac{\log x}{x^2}$$

   defined for $x > 0$ are slowly varying at infinity?

2. Let $\tau = \inf\{t > 0 : B_t = 1\}$ be the hitting time of level 1 by the standard Brownian motion B_t which starts from $B_0 = 0$. Show that τ has the Lévy distribution, that is, its density is $(2\pi y^3)^{-1/2}\exp(-1/(2y))$ for $y > 0$.

   Hint: It can be used without proof that as a consequence of the reflection principle

$$\mathbf{P}\left(\max\{B_s : s \in [0,t]\} > x\right) = 2\left(1 - \Phi\left(\frac{x}{\sqrt{t}}\right)\right)$$

   where $\Phi(x) = \int_{-\infty}^{x}(2\pi)^{-1/2}\exp(-y^2/2)\,\mathrm{d}y$ is the standard normal distribution function. The distribution function of τ can be directly expressed by the probability in the equality above.

3. Let $X_1, X_2, \ldots$ be an iid. sequence of negative binomial random variables with parameters $p \in (0,1)$ and $m = 2$, that is,

$$\mathbf{P}(X = k) = (k+1)p^2(1-p)^k$$

   for $k = 0, 1, 2, \ldots$ and let $M_n = \max(X_1, \ldots, X_n)$. Show that there is no such normalization under which the sequence of maxima M_n has a non-degenerate limit law.

   Hint: Show that the necessary condition

$$\lim_{x \uparrow x_F} \frac{\overline{F}(x+)}{\overline{F}(x)} = 1$$

   fails to hold along a sequence of integers where $\overline{F}(x) = 1 - F(x)$ is the tail probability function of X_i. More precisely $\overline{F}(k+1)/\overline{F}(k)$ converges to $1 - p$ for the integers $k \to \infty$.

4. Let $U_1, U_2, \ldots$ be a sequence of independent and identically distributed uniform random variables on $[0,1]$. Define $X_n = e^{U_n}$ for all $n = 1, 2, \ldots$ and let $M_n = \max(X_1, \ldots, X_n)$. Which non-trivial limit distribution does the renormalized sequence of M_n converge to? Under what normalization?

5. Let $X_1, X_2, \ldots$ be random variables defined on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$ which have the same distribution function $F(x)$ but we do not assume anything about their dependence structure. Let $M_n := \max_{1 \le i \le n} |X_i|$. Suppose that there is an $s > 0$ such that $\int_{-\infty}^{\infty} e^{s|x|}\mathrm{d}F(x) < \infty$, that is, $\mathbf{E}(e^{s|X_i|}) < \infty$. Prove that for any sequence b_n that goes to ∞ and for fixed $\delta > 0$,

$$\lim_{n\to\infty} \mathbf{P}\left((b_n \log n)^{-1}|M_n| > \delta\right) = 0.$$

   Hint: Use the following inequality

$$\mathbf{P}\left(\max_{1\le i\le n} |X_i| > \lambda\right) = \mathbf{P}(\cup_{i=1}^{n}\{|X_i| > \lambda\}) \le \sum_{i=1}^{n} \mathbf{P}(|X_i| > \lambda) = n\mathbf{P}(|X_i| > \lambda)$$

   and a Markov type inequality.