Superdiffusive elephant random walk with general step distribution

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Outline

- Elephant random walk, asymptotics and history
- General step distribution and results
- Limiting moments
- Convergence in L^p

Introduction

joint work with József Kiss

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Elephant random walk

nearest neighbour random walk X_n on $\mathbb{Z}$ with memory

parameters: $p, q \in [0,1]$

Steps:

$\mathrm{P}(X_1 = 1) = q$, $\mathrm{P}(X_1 = -1) = 1 - q$

$$X_{n+1} = \begin{cases} X_K & \text{with probability} \quad p \\ -X_K & \text{with probability} \quad 1-p \end{cases}$$

where K is uniform random from $\{1, 2, \ldots, n\}$ independent from the past

Displacement:

$$S_n = X_1 + X_2 + \cdots + X_n$$

Connection to Pólya-type urns

Urn with balls of two colors: rose and lemon
Their number after n steps: R_n and L_n

Initially in the urn: a single rose ball with probability q, a single lemon ball with probability $1-q$

Dynamics:

- a ball is chosen uniformly from the urn
- the chosen ball is returned to the urn
  plus a ball of the same color with probability p
  or a ball of the other color with probability $1-p$

Lemma

The process $R_n - L_n$ has the same law as the elephant random walk S_n.

R_n: steps to the right
L_n: steps to the left

Meaning of parameters

1. q initial bias, $\mathrm{P}(X_1 = 1) = q$, $\mathrm{P}(X_1 = -1) = 1 - q$
2. p memory parameter,

$$X_{n+1} = \begin{cases} X_K & \text{with probability} \quad p \\ -X_K & \text{with probability} \quad 1-p \end{cases}$$

where K is uniform from $\{1, 2, \ldots, n\}$ independent from the past

Lemma

Let $X_n^{(q)}$ denote the elephant random walk with initial bias q. Then

$$(X_n^{(q)})_n \stackrel{\mathrm{d}}{=} (\chi_q X_n^{(1)})_n$$

where χ_q is independent of $(X_n^{(1)})$ and $\mathrm{P}(\chi_q = 1) = 1 - \mathrm{P}(\chi_q = -1) = q$.

Role of q is marginal, we may assume $q = 1/2$.

Equivalent description

We give a natural alternative definition for initial bias $q = 1/2$.

Let $\xi_1, \xi_2, \dots$ be i.i.d. with $\mathrm{P}(\xi_1 = 1) = \mathrm{P}(\xi_1 = -1) = 1/2$.
Define $X_1 = \xi_1$ and let

$$X_{n+1} = \begin{cases} X_K & \text{with probability} \quad \alpha \\ \xi_{n+1} & \text{with probability} \quad 1-\alpha \end{cases}$$

where K is a uniform random from $\{1, 2, \dots, n\}$ independent from the past.
Let $S_n = X_1 + \cdots + X_n$.

Lemma

The elephant random walk process defined with $p \in [1/2, 1]$ and $q = 1/2$ has the same law as the one defined above with

$$\alpha = 2p - 1 \in [0, 1].$$

Asymptotic behaviour

Theorem

1. *Diffusive regime:* $\alpha < 1/2$:

$$\frac{S_n}{\sqrt{n}} \overset{\mathrm{d}}{\Rightarrow} \mathcal{N}\left(0, \frac{1}{1-2\alpha}\right)$$

2. *Critical regime:* $\alpha = 1/2$:

$$\frac{S_n}{\sqrt{n \log n}} \overset{\mathrm{d}}{\Rightarrow} \mathcal{N}(0,1)$$

3. *Superdiffusive regime:* $\alpha > 1/2$:

$$\frac{S_n}{n^\alpha} \to Q$$

almost surely and in L^p for all $p \geq 1$ as $n \to \infty$ where Q is a non-degenerate random variable with law depending on α.

Martingales

Let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Then by definition

$$\mathrm{E}(X_{n+1}|\mathcal{F}_n) = \frac{\alpha}{n}X_1 + \cdots + \frac{\alpha}{n}X_n = \frac{\alpha}{n}S_n$$

meaning that

$$\mathrm{E}(S_{n+1}|\mathcal{F}_n) = \left(1 + \frac{\alpha}{n}\right) S_n.$$

Hence

$$Q_n = a_n S_n$$

is a martingale where

$$a_n = \Gamma(1+\alpha)^{-1} \prod_{k=1}^{n-1} \left(1 + \frac{\alpha}{k}\right)^{-1} = \frac{\Gamma(n)}{\Gamma(n+\alpha)} \sim n^{-\alpha}$$

as $n \to \infty$.

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Quadratic variation

The martingale $Q_n = a_n S_n$ where $a_n = \Gamma(1+\alpha)^{-1} \prod_{k=1}^{n-1} \left(1+\frac{\alpha}{k}\right)^{-1} \sim n^{-\alpha}$ can be written as

$$Q_n = \sum_{k=1}^{n} a_k \varepsilon_k$$

where

$$\varepsilon_n = \frac{1}{a_n}(Q_n - Q_{n-1}) = S_n - \frac{a_{n-1}}{a_n} S_{n-1} = S_n - \mathrm{E}(S_n | \mathcal{F}_{n-1}) = X_n - \mathrm{E}(X_n | \mathcal{F}_{n-1})$$

The predictable quadratic variation

$$\langle Q \rangle_n \sim \sum_{k=1}^{n} a_k^2 \sim \sum_{k=1}^{n} k^{-2\alpha}.$$

1. Diffusive regime: $\alpha < 1/2$: $\langle Q \rangle_n \sim c n^{1-2\alpha}$, martingale CLT implies $Q_n / \sqrt{\langle Q \rangle_n} \sim S_n / \sqrt{n}$ converges to normal
2. Critical regime: $\alpha = 1/2$: $\langle Q \rangle_n \sim c \log n$, $S_n / \sqrt{n \log n}$ goes to normal
3. Superdiffusive regime: $\alpha > 1/2$: $\langle Q \rangle_\infty = \lim_{n\to\infty} \langle Q \rangle_n < \infty$, $Q_n \sim S_n / n^\alpha$ converges

History

- Drezner, Farnum, 1993: correlated Bernoulli process, time-dependent memory parameter
- Heyde, 2004: phase transition in the correlated Bernoulli process, non-Gaussian limit conjectured, natural martingales
- Schütz, Trimper, 2004: elephant random walk
- Baur, Bertoin, 2016: connection to Pólya-type urns
- Bercu, 2017: non-Gaussian limit rigorously, almost sure behaviour using martingales, limiting moments in the superdiffusive regime
- Businger, 2018: shark random swim: heavy-tailed steps
- Bertoin, 2022: zeros of the elephant random walk
- Kiss, V., 2022: limiting superdiffusive moments for general steps
- Guérin, Laulin, Raschel, 2023: fixed-point equation for the limiting distribution in the superdiffusive regime

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General step distribution

Let $\xi_1, \xi_2, \dots$ be i.i.d. random variables.
Define $X_1 = \xi_1$ and let

$$X_{n+1} = \begin{cases} X_K & \text{with probability} \quad \alpha \\ \xi_{n+1} & \text{with probability} \quad 1-\alpha \end{cases}$$

where K is a uniform random from $\{1, 2, \dots, n\}$ independent from the past.
Let $S_n = X_1 + \cdots + X_n$ as before.
Let

$$m_k = \mathrm{E}(\xi_1^k), \quad M_k = \mathrm{E}((\xi_1 - m_1)^k)$$

be the kth moments and centered moments.

Note that all steps have the same distribution, in particular
$\mathrm{E}((X_n - m_1)^k) = M_k$.

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Results

Theorem (J. Kiss, B. V., 2022)

Let $\alpha \in (1/2, 1]$*.*

1. *If* $m_2 < \infty$*, then*
$$\frac{S_n - nm_1}{n^\alpha} \to Q \tag{1}$$
almost surely as $n \to \infty$ *where* Q *is a non-degenerate random variable.*
2. *Assume that* $m_p < \infty$ *for some positive even integer. Then* (1) *holds also in* L^p*.*
3. *If* $m_4 < \infty$*, then*
$$\begin{aligned} \mathrm{E}(Q) &= 0, \\ \mathrm{E}\left(Q^2\right) &= \frac{M_2}{(2\alpha-1)\Gamma(2\alpha)}, \\ \mathrm{E}\left(Q^3\right) &= \frac{4M_3}{(3\alpha-1)\Gamma(3\alpha)}, \\ \mathrm{E}\left(Q^4\right) &= \frac{6(3(2\alpha-1)^2 M_4 + 2(1-\alpha)(5\alpha-2)M_2^2)}{(2\alpha-1)^2(4\alpha-1)\Gamma(4\alpha)}. \end{aligned}$$

Limiting moments

Proof idea: to deduce recursions for the moments of S_n.
Assume centered steps ($m_1 = 0$) for this proof.
We write $S_{n+1} = S_n + X_{n+1}$ and we compute

$$\mathrm{E}(S_{n+1}^2 | \mathcal{F}_n) = \mathrm{E}(S_n^2 + 2S_n X_{n+1} + X_{n+1}^2 | \mathcal{F}_n).$$

Here

$$\mathrm{E}(X_{n+1} | \mathcal{F}_n) = \frac{\alpha}{n} S_n$$

and

$$\mathrm{E}(X_{n+1}^2 | \mathcal{F}_n) = \frac{\alpha}{n} \sum_{k=1}^{n} X_k^2 + (1-\alpha)\mathrm{E}(\xi_{n+1}^2)$$

which does not simplify for general step distribution.
Hence

$$\mathrm{E}(S_{n+1}^2) = \left(1 + \frac{2\alpha}{n}\right) \mathrm{E}(S_n^2) + M_2.$$

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Third and higher moment

For the recursion on the third moment, we write

$$\mathrm{E}(S_{n+1}^3|\mathcal{F}_n) = \mathrm{E}(S_n^3 + 3S_n^2 X_{n+1} + 3S_n X_{n+1}^2 + X_{n+1}^3|\mathcal{F}_n).$$

Most interestingly,

$$\begin{aligned}\mathrm{E}(S_n X_{n+1}^2|\mathcal{F}_n) = S_n \mathrm{E}(X_{n+1}^2|\mathcal{F}_n) = S_n \left(\frac{\alpha}{n}\sum_{k=1}^n X_k^2 + (1-\alpha)\mathrm{E}(\xi_{n+1}^2)\right) \\ = \frac{\alpha}{n} S_n T_n + S_n m_2\end{aligned}$$

where $T_n = \sum_{k=1}^n (X_k^2 - m_2)$. By taking expectation

$$\mathrm{E}(S_{n+1}^3) = \left(1 + \frac{3\alpha}{n}\right)\mathrm{E}(S_n^3) + \frac{3\alpha}{n}\mathrm{E}(S_n T_n) + M_3$$

which requires a recursion for $\mathrm{E}(S_n T_n)$.

For the recursion on $\mathrm{E}(S_n^4)$ one uses
$\mathrm{E}(S_n^2 T_n)$, $\mathrm{E}(S_n U_n)$, $\mathrm{E}(T_n^2)$ where $U_n = \sum_{k=1}^n (X_k^3 - m_3)$.

Convergence in L^p

We prove that the martingale Q_n is bounded in L^p by a recursion for $\mathrm{E}(Q_n^p)$.

We write $Q_{n+1} = Q_n + a_{n+1}\varepsilon_{n+1}$ where $\varepsilon_{n+1} = X_{n+1} - \mathrm{E}(X_{n+1}|\mathcal{F}_n)$ and we expand the conditional expectation

$$\mathrm{E}(Q_{n+1}^p|\mathcal{F}_n) = \sum_{k=0}^{p} \binom{p}{k} a_{n+1}^k \mathrm{E}(\varepsilon_{n+1}^k|\mathcal{F}_n) Q_n^{p-k}$$

$k=0$ term: Q_n^p,
$k=1$ term: 0 because $\mathrm{E}(\varepsilon_{n+1}|\mathcal{F}_n)=0$.

The expectation of remaining terms can be bounded using Hölder's inequality

$$|\mathrm{E}(\mathrm{E}(\varepsilon_{n+1}^k|\mathcal{F}_n)Q_n^{p-k})| \leq (\mathrm{E}(\varepsilon_{n+1}^p))^{k/p}(\mathrm{E}(Q_n^p))^{(p-k)/p}.$$

The exponents $k/p, (p-k)/p \leq 1$ and p is an even integer, hence

$$\mathrm{E}(Q_{n+1}^p) \leq \left(1 + a_{n+1}^2 2^p (1 + \mathrm{E}(\varepsilon_{n+1}^p))\right) \mathrm{E}(Q_n^p) + a_{n+1}^2 2^p (1 + \mathrm{E}(\varepsilon_{n+1}^p)).$$

Convergence in L^p

We have the recursive inequality

$$\mathrm{E}(Q_{n+1}^p) \le \left(1 + a_{n+1}^2 2^p (1 + \mathrm{E}(\varepsilon_{n+1}^p))\right) \mathrm{E}(Q_n^p) + a_{n+1}^2 2^p (1 + \mathrm{E}(\varepsilon_{n+1}^p))$$

where $\varepsilon_{n+1} = X_{n+1} - \mathrm{E}(X_{n+1} | \mathcal{F}_n)$ hence $\mathrm{E}(\varepsilon_{n+1}^p) \le 2^p m_p$.

Lemma

If the real positive sequence b_n satisfies

$$b_{n+1} \le \left(1 + \frac{c}{n^\beta}\right) b_n + \frac{c}{n^\beta}$$

for some $\beta > 1$ and $c > 0$ and $b_1 \le c$, then b_n remains bounded in n.

Remember $a_n \sim n^{-\alpha}$ and $\alpha > 1/2$ in the superdiffusive regime.
By the lemma with $\beta = 2\alpha > 1$, the martingale Q_n remains bounded in L^p, hence $Q_n = S_n / n^\alpha$ converges in L^p.

The end

Thank you for your attention!