Stochastic processes exam

16th Dec 2024

Theoretical part

1. (a) (2 points) Define birth and death chains in discrete time on finite state space.
   (b) (2+5 points) State and prove the reversibility of birth and death chains in discrete time on finite state space by expressing the stationary distribution in terms of the transition probabilities.
2. (2+2+5 points) What is the exit distribution of a continuous-time Markov chain on finite state space? Write down the setup of the problem. State and prove the equations the exit probabilities satisfy.
3. (a) (3 points) Define the conditional expectation of a random variable with respect to a σ-algebra.
   (b) (2 points) State the Radon–Nikodym theorem without proof.
   (c) (4 points) Show the existence and uniqueness of the conditional expectation using the Radon–Nikodym theorem.

Exercise part

4. (3+2+2 points) Consider the Markov chain on $S = \{1, 2, 3, 4\}$ with transition matrix

$$P = \begin{pmatrix} 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0.3 & 0.7 \\ 0.8 & 0.2 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \end{pmatrix}.$$

   (a) Write down how to determine the stationary distribution of P. The answer does not have to be computed explicitly but it can be expressed with the inverse of an explicit matrix.
   (b) Compute P^2 and all stationary distributions of P^2.
   (c) Find the limit of $P^{2n}(x, x)$ as $n \to \infty$.

5. (5+2 points) Consider a two station queueing network in which arrivals only occur at the first server and do so at rate 1. If a customer finds server 1 free he enters the system; otherwise he goes away. When a customer is done at the first server he moves on to the second server if it is free and leaves the system if it is not. Suppose that server 1 serves at rate 3 while server 2 serves at rate 2. Formulate a Markov chain model for this system with state space $\{0, 1, 2, 12\}$ where the state indicates the servers who are busy.
   (a) In the long run what proportion of customers enter the system?
   (b) What proportion of the customers visit server 2?
6. (3+4 points) Let $S_n = X_1 + \cdots + X_n$ where $X_1, X_2, \ldots$ are independent with $\mathbf{E}(X_i) = 0$ and $\mathrm{Var}(X_i) = \sigma^2$. Show that $M_n = S_n^2 - n\sigma^2$ is a martingale. Let $\tau = \min\{n : |S_n| > a\}$. By computing the expectation of the stopped martingale $M_{n\wedge\tau}$ show that $\mathbf{E}(\tau) \geq a^2/\sigma^2$.
7. (7 points) Let $B(t)$ be a Brownian motion and fix $s > 0$. Prove that $B(t + s) - B(s)$ is also a Brownian motion.