Stochastic processes exam

10th Jan 2025

Theoretical part

1. (a) (2 points) Define the stationary distribution of the discrete time Markov chain X_n.
   (b) (2 points) Assume that the Markov chain is irreducible. Write down those equations for the components of the stationary distribution vector in terms of the transition matrix which determine the stationary distribution.
   (c) (2+3 points) Explain and prove how the stationary distribution of an irreducible Markov chain can be computed using the inverse of a certain matrix.
2. (a) (2 points) Define the (time homogeneous) Poisson process of parameter $\lambda > 0$ on $\mathbb{R}_+$.
   (b) (2 points) We thin the Poisson process of parameter λ with probability p, that is, we keep each point with probability p independently for the points. What is the distribution of the resulting set of points? (No proof is needed here.)
   (c) (2+3 points) What is the distribution of the first point of a Poisson process of parameter λ conditionally given that there is exactly one point in the interval $[0, t]$ for some fixed $t > 0$. (The answer has to be proven.)
3. Let M_n be a martingale and let H_n be a predictable betting strategy.
   (a) (2 points) Define the predictability of H_n.
   (b) (2 points) Define the wealth W_n after step n using the martingale M_n and the betting strategy H_n started from initial wealth W_0.
   (c) (2+3 points) State and prove a sufficient condition under which the wealth process W_n is a martingale if M_n is a martingale.

Exercise part

4. (5+2 points) A knight can move two squares horizontally and one square vertically or two squares vertically and one horizontally in one step. Let X_n be the sequence of squares that results if we pick one of the knight's legal moves at random independently in each step.
   (a) Find the stationary distribution.
   (b) What is the expected number of moves to return to the corner $(1, 1)$ when we start there.
5. (3+4 points) Consider two machines that are maintained by a single repairman. Machine i functions for an exponentially distributed amount of time with rate λ_i before it fails. The repair times for each unit are exponential with rate μ_i. They are repaired in the order in which they fail.
   (a) Formulate a Markov chain model for this situation with state space $\{0, 1, 2, 12, 21\}$.
   (b) Suppose that $\lambda_1 = 1$, $\mu_1 = 2$, $\lambda_2 = 3$, $\mu_2 = 4$. Find the stationary distribution.
6. (2+3+2 points) Let $U_1, U_2, \ldots$ be i.i.d. random variables on $(0, 1)$ with density $f(x) = 2x$ and define $Y_0 = 1$ and $Y_n = U_n U_{n-1} \ldots U_0$.
   (a) Show that $M_n = (3/2)^n Y_n$ is a martingale.
   (b) Use the fact that $\log Y_n = \log U_1 + \cdots + \log U_n$ to show that almost surely
   $$\frac{\log Y_n}{n} \to -\frac{1}{2}.$$
   (c) Use (b) and the fact that $\log(3/2) - 1/2 < 0$ to conclude $M_n \to 0$, i.e., in this "fair" game our fortune always converges to 0 as time tends to infinity.
7. (7 points) Let $B(t)$ and $\widetilde{B}(t)$ be two independent Brownian motions. Prove that
   $$\frac{1}{2}B(t) - \frac{\sqrt{3}}{2}\widetilde{B}(t)$$
   is also a Brownian motion.