Stochastic processes exam

24th Jan 2025

Theoretical part

1. (a) (2 points) Define the branching processes as discrete time Markov chains.
   (b) (2 points) Define the generating function of the probability distribution on the non-negative integers given by the probabilities $(p_k)_{k=0}^{\infty}$.
   (c) (2+3 points) State and prove how the extinction probability of a branching process can be determined using the generating function g of the offspring distribution.
   The lemma about the generating function of the sum of i.i.d. random variables with a random number of summands can be used without proof.

2. (3+6 points) State and prove Kolmogorov's forward differential equation for continuous time Markov chains on finite state space.

3. (a) (3 points) Define the upcrossings of an interval $[a, b]$ by the stochastic process X_n.
   (b) (6 points) Prove that a supermartingale can only have finitely many upcrossings of any interval $[a, b]$. Doob's (super)martingale inequality can be used without proof.

Exercise part

4. (7 points) Consider the Markov chain with state space $\{0, 1, 2, \dots\}$ and transition probabilities

$$p(m, m+1) = \frac{1}{2}\left(1 - \frac{1}{m+3}\right) \qquad \text{for } m \geq 0$$
$$p(m, m-1) = \frac{1}{2}\left(1 + \frac{1}{m+3}\right) \qquad \text{for } m \geq 1$$

   and $p(0, 0) = 1 - p(0, 1) = 2/3$. Find the stationary distribution π.

5. (7 points) A math professor waits at the bus stop at the Mittag–Leffler Institute in the suburbs of Stockholm, Sweden. Since he has forgotten to find out about the bus schedule, his waiting time until the next bus is uniform on $(0, 1)$. Cars drive by the bus stop at rate 12 per hour. Each will take him into town with probability $1/4$. What is the probability he will end up riding the bus?

6. (5+2 points) A taxi company has three cabs. Calls come in to the dispatcher at times of a Poisson process with rate 3 per hour. Suppose that each requires an exponential amount of time with mean 30 minutes, and that callers will hang up if they hear there are no cabs available.
   (a) What is the probability all three cabs are busy when a call comes in?
   (b) In the long run, on the average how many customers are served per hour?

7. (7 points) Let $B(t)$ be the one-dimensional Brownian motion. Consider the stochastic process $V(0) = 0$ and $V(t) = tB(1/t)$. Prove that $V(t)$ is also a Brownian motion.