Applications of Stochastics — Exercise sheet 2

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From **Sheet 1**, we have not done Exercises 4, 5(b), and 7(c)(d)(e). You should do them now. Please note that 7(c) had a typo: it should be $X_t^2 - t$ instead of $X_t - t^2$ (now corrected there).

▷ **Exercise 1** (Discrete Chung-Fuchs theorem).

(a) Let $S_n = X_1 + \cdots + X_n$ be a random walk with i.i.d. jumps in $\mathbb{Z}^d$. Show that, for any $m \in \mathbb{Z}_+$,

$$\sum_{n=0}^{\infty} \mathbf{P}\big[\, \|S_n\|_\infty \leq m \,\big] \leq (2m+1)^d \sum_{n=0}^{\infty} \mathbf{P}[\, S_n = \underline{0} \,] .$$

(Hint: for any $v \in \mathbb{Z}^d$ with $\|v\|_\infty \leq m$, the event $\{S_n = v\}$ can be decomposed as $\bigcup_{\ell=0}^{n} \{S_n = v,\ T_v = \ell\}$, according to the first hitting time of v.)

(b) Assume now that $d = 1$, and that S_n satisfies the following Weak Law of Large Numbers: $S_n/n \xrightarrow{p} 0$. Notice that, for any $m \in \mathbb{Z}_+$ and $A > 0$, part (a) implies

$$\sum_{n=0}^{\infty} \mathbf{P}[\, S_n = 0 \,] \geq \frac{1}{2m+1} \sum_{n=0}^{\infty} \mathbf{P}\big[\, |S_n| \leq m \,\big] \geq \frac{1}{2m+1} \sum_{n=0}^{\lfloor Am \rfloor} \mathbf{P}\big[\, |S_n| \leq n/A \,\big] .$$

Deduce from this and the WLLN that the expected number of returns to 0 is infinite. Conclude that the walk is recurrent.

▷ **Exercise 2.** Show that if $\{M_i\}_{i=0}^{\infty}$ is a martingale, then the differences $X_i = M_i - M_{i-1}$ satisfy the uncorrelatedness condition $\mathbf{E}\big[\, X_{i_1} \cdots X_{i_k} \,\big] = 0$, for any $k \in \mathbb{Z}_+$ and $i_1 < i_2 < \cdots < i_k$.

▷ **Exercise 3.** Using the exponential Markov inequality as in class, together with the moment generating function $m_X(t) = \mathbf{E}\big[\, e^{tX} \,\big]$, prove the following two exponential concentration inequalities:

(a) If $S_n = X_1 + \cdots + X_n$ is a sum of i.i.d. variables with $\mathbf{E}X_i = \mu$ and $m_X(t_0) < \infty$ for some $t_0 > 0$, then, for any $\delta > 0$ there exist $c_\delta > 0$ and $C_\delta < \infty$ (which also depend on the distribution of X_i) such that

$$\mathbf{P}\big[\, |S_n/n - \mu| > \delta \,\big] < C_\delta \, e^{-c_\delta n},$$

for any n. (Hint: use that $\frac{d}{dt} \log m_X(t)\big|_{t=0} = 0$, while $\frac{d}{dt} \delta t\big|_{t=0} > 0$.)

(b) For any $\delta > 0$ there exist $c_\delta > 0$ and $C_\delta < \infty$ such that

$$\mathbf{P}\big[\, |\mathsf{Poi}(\lambda) - \lambda| > \delta\lambda \,\big] < C_\delta \, e^{-c_\delta \lambda},$$

for any $\lambda > 0$. (Hint: we know what the exponential generating function of $\mathsf{Poi}(\lambda)$ is.)

▷ **Exercise 4.** Let $X_k(n)$ be the number of degree k vertices in the Erdős-Rényi random graph $G(n, \lambda/n)$, with any $\lambda \in \mathbb{R}_+$ fixed. Show that $X_k(n)/n$ converges in probability, as $n \to \infty$, to $\mathbf{P}[\, \mathsf{Poisson}(\lambda) = k \,]$. (Hint: the 1st moment of $X_k(n)$ is clear; then use the 2nd moment method.)