Applications of Stochastics — Exercise sheet 4

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▷ **Exercise 1.** As in class, let $\nu_1, \nu_2, \ldots$ be positive reals satisfying the recursion

$$\nu_{t+1} = \left(1 - \frac{\alpha_t}{t}\right)\nu_t + \frac{\beta_t}{t},$$

where α_t, β_t are positive reals converging to some positive α and β, respectively. Find $\lim_{t\to\infty} \nu_t$.

▷ **Exercise 2.** Let X_1 be ± 1 with probability $1/2$ each, and let $X_2, \ldots, X_n$ be X_1. (Not just in distribution, but really.) Then $S_k = X_1 + \cdots + X_k$ for $k = 0, 1, \ldots, n-1$ satisfy $|S_{k+1} - S_k| = 1$, and we have $\mathbf{E}S_n = 0$, but S_n is not at all concentrated around its mean. (Right?) Why does the Doob martingale $M_k := \mathbf{E}[\, S_n \mid S_0, S_1, \ldots, S_k \,]$ together with Azuma-Hoeffding *not* prove concentration for $M_n = S_n$? (This is a somewhat silly exercise, but I saw it on mathoverflow.net, so have decided to pose it for you.)

It is very strange to use the Strong Law of Large Numbers and in general to talk about almost sure convergence without you having ever learnt anything about it, so here is the fundamental lemma:

Lemma 1 (Borel-Cantelli lemma)**.** *If $A_1, A_2, \ldots$ are events in a probability space with $\sum_{n=1}^{\infty} \mathbf{P}[\, A_n \,] < \infty$, then*

$$\mathbf{P}\Big[\bigcap_{N\geq 1} \bigcup_{n \geq N} A_n \Big] = \mathbf{P}[\, A_n \text{ occurs infinitely often} \,] = 0\,.$$

Proof. Let $X = \#\{n \geq 1 : A_n \text{ is satisfied}\}$. Then $\mathbf{E}X = \sum_{n=1}^{\infty} \mathbf{P}[\, A_n \,]$ by Fubini. Since $\mathbf{E}X < \infty$, we have $\mathbf{P}[\, X = \infty \,] = 0$. □

Part (c) of the next exercise was mentioned in a more general version in class, but without a proof. The task is to do the proof in the special case of $\mathsf{Expon}(1/\mu)$: this will require a bit of thinking then an application of Borel-Cantelli.

▷ **Exercise 3.** As in class, let $\xi_1, \xi_2, \ldots$ be the i.i.d. lifetimes of the light bulbs, with $\mathbf{E}\xi_i = \mu \in (0, \infty]$, and we have a janitor who visits the corridor at times given by a Poisson process with intensity λ, and if he sees that the bulb is dead, he replaces it by a new one. Thus the times $\tau_1, \tau_2, \ldots$ passing between the death of a light bulb and the next visit of the janitor are i.i.d. $\mathsf{Expon}(\lambda)$ variables.

(a) At what rate are bulbs replaced?

(b) What is the almost sure limiting fraction of visits by the janitor on which the bulb is working?

(c) Now assume that $\xi_i \sim \mathsf{Expon}(1/\mu)$. What is the limiting fraction of time that the light works?