Applications of Stochastics — Exercise sheet 7

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To make sure you understand what the measurability of having an infinite cluster means:

▷ **Exercise 1.** Let $G(V,E)$ be any bounded degree infinite graph, and $S_n \nearrow V$ an exhaustion by finite connected subsets. Is it true that, for $p > p_c(G)$, we have

$$\lim_{n\to\infty} \mathbf{P}_p[\,\text{largest cluster for percolation inside } S_n \text{ is the subset of an infinite cluster}\,] = 1\,?$$

Generalizations of the basic arguments from class for bounds on $p_c(G) = p_c(G, \text{bond})$:

▷ **Exercise 2.**

(a) Show that in any graph $G(V,E)$ with maximal degree Δ, we have $p_c(G) \geq 1/(\Delta-1)$.

(b) Show that if in a graph G the number of minimal edge-cutsets (a subset of edges whose removal disconnects a given vertex from infinity, minimal w.r.t. containment) of size n is at most $\exp(Cn)$ for some $C < \infty$, then $p_c(G) \leq 1 - \epsilon(C) < 1$.

(c) Fix $o \in V(G)$ in a graph with maximal degree Δ. Prove that the number of connected sets $o \in S \subset V(G)$ of size n is at most $\Delta(\Delta-1)^{2n-3}$. (Hint: any S has a spanning tree, and one can go around a tree visiting each edge twice.) Conclude that $\mathbb{Z}^d$, $d \geq 2$, has an exponential bound on the number of minimal cutsets. In particular, $p_c(\mathbb{Z}^d) < 1$, although we already knew that from $\mathbb{Z}^2 \subseteq \mathbb{Z}^d$.

The next one is a bit more challenging, but still not that hard:

▷ **Exercise 3.** Show that, for any infinite graph $G(V,E)$ with finite degrees, $p_c(G, \text{bond}) \leq p_c(G, \text{site})$. (Hint: explore the site percolation configuration ξ in a way that gives you a coupling with a bond percolation ω, such that whenever the cluster of some $o \in V(G)$ is infinite in ξ, it will also be infinite in ω.)

Two exercises on the Harris-FKG inequality:

▷ **Exercise 4.** Consider a product probability measure $\mu_1 \otimes \cdots \otimes \mu_n$ on $\mathbb{R}^n$. Let $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be two square-integrable monotone increasing functions (i.e., if $x_i \leq y_i$ for all $i = 1, \dots, n$, then $f(x_1, \dots, x_n) \leq f(y_1, \dots, y_n)$ holds). The Harris-FKG inequality says that f and g are then positively correlated:

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1,\dots,x_n)\, g(x_1,\dots,x_n)\, d\mu_1(x_1) \dots d\mu(x_n) \geq$$
$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1,\dots,x_n)\, d\mu_1(x_1) \dots d\mu(x_n) \times \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g(x_1,\dots,x_n)\, d\mu_1(x_1) \dots d\mu(x_n)\,.$$

We proved this for $n = 1$ in class. Prove the full statement by induction on n.

▷ **Exercise 5.** Show that the "conditional FKG-inequality" does not hold: find three increasing events A, B, C in some $\mathsf{Ber}(p)$ product measure space such that $\mathbf{P}_p[\,AB \mid C\,] < \mathbf{P}_p[\,A \mid C\,]\,\mathbf{P}_p[\,B \mid C\,]$.

Two bonus exercises on critical percolation on trees:

▷ **Exercise 6.** * Find the critical percolation density p_c of the following two trees (see Figure 1):

(a) The quasi-transitive tree with degree 3 and degree 2 vertices alternating.

(b) The so-called 3-1-tree, which has 2^n vertices on each level n, with the left 2^{n-1} vertices each having one child, the right 2^{n-1} vertices each having three children; the root has two children.

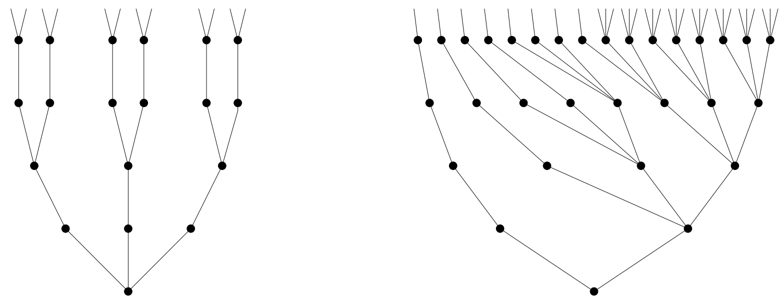

Figure 1: A quasi-transitive tree and the 3-1 tree.

▷ **Exercise 7.** * Consider a spherically symmetric tree T where each vertex on the n^{th} level T_n has $d_n \in \{k, k+1\}$ children, in such a way that $\lim_{n\to\infty} |T_n|^{1/n} = k$, but $\sum_{n=0}^{\infty} k^n/|T_n| < \infty$. Using the second moment method, show that $p_c = 1/k$ and $\theta(p_c) > 0$.