Applications of Stochastics — Exercise sheet 9

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We used part (a) of the first exercise to understand the basic behaviour of G/G/1 queuing systems with $\lambda < \mu$.

▷ **Exercise 1.** Consider a random walk $S_n = X_1 + \cdots + X_n$ on $\mathbb{R}$, with iid increments satisfying $\mathbf{E}X_i < 0$. Let $S_{\max} = \max\{S_0, S_1, S_2, \dots\}$, an almost surely finite variable, since $S_n \to -\infty$.

(a) Let $N := \min\{n > 0 : S_n < 0\}$. Show that $\mathbf{E}N < \infty$. (Hint: let $M \geq 0$ be the largest m when $S_m = S_{\max}$, and look at the two pieces of trajectories $S_M, S_{M-1}, \dots, S_0$ and $S_M, S_{M+1}, \dots$ separately.)

(b) Assume that $\mathbf{E}\big[\, e^{tX_i} \,\big] < \infty$ for some $t > 0$. Show that $\mathbf{P}\big[\, S_{\max} > m \,\big] < C\exp(-c\sqrt{m})$ for some $0 < c, C < \infty$, hence $\mathbf{E}S_{\max} < \infty$.

(c) Now assume $\mathbf{E}(X_i \vee 0)^2 < \infty$ only. Show that $\mathbf{E}S_T < \infty$, where $T := \min\{n > 0 : S_n > 0\}$, and conclude that $\mathbf{E}S_{\max} < \infty$ still holds. (Hint: for simplicity, assume that X_i is integer valued. Now estimate $\mathbf{P}[\, S_T = k \,]$ using a decomposition according to the possible values of $T - 1 = n$ and $S_{T-1} = -\ell$, and using that $\sum_{n\geq 1} \mathbf{P}[\, S_n = -\ell \,] < C < \infty$, uniformly in $\ell > 0$.)

(d) * Assume now that $\mathbf{E}(X_i \vee 0)^2 = \infty$. Is it true that $\mathbf{E}S_T = \infty$?

As in class, consider a **G/G/1 queuing system**, with iid inter-arrival times $A_1, A_2, \dots$ of mean $1/\lambda$, iid service times $B_1, B_2, \dots$ of mean $1/\mu$, with $\lambda < \mu$, the first customer arriving at time 0 at an empty system. Let W_n be the time the nth customer has to wait for her service to start. Recall or observe that

$$W_{n+1} = (W_n + B_n - A_{n+1}) \vee 0\,,$$

and let $\overline{W} := \mathbf{E}\big[\, \lim_{n\to\infty} W_n \,\big]$, which exists and is finite by the previous exercise.

Let $Q_t := \#\{\text{people in the queue at time } t\}$ and $Q_t^+ := \#\{\text{people in the system at time } t\}$. We accepted without a proof that the average long-term queue size $\overline{Q} := \lim_{t\to\infty} \frac{1}{t}\int_0^t Q_s\, ds$ exists and is non-random; similarly for $\overline{Q^+}$. We also stated **Little's law**, proved in a very hand-waving manner:

$$\overline{Q} = \lambda\, \overline{W} \qquad \text{and} \qquad \overline{Q^+} = \lambda\, (\overline{W} + \overline{B})\,,$$

where $\overline{B} = \mathbf{E}[\, B_1 \,] = \frac{1}{\mu}$. Subtracting the first Little identity from the second one, we obtained that the system utilization ratio is $\rho := \lim_{t\to\infty} \mathbf{P}[\, Q_t^+ > 0 \,] = \frac{\lambda}{\mu}$.

▷ **Exercise 2.** Specialize to the case of **M/G/1 systems**. Let $H_1 := \inf\{t > 0 : Q_t^+ = 0\}$ be the length of the first busy period. For the the Laplace transform $h(s) := \mathbf{E}[\, e^{-sH_1} \,]$, we showed in class that

$$h(s) = b(s + \lambda - \lambda h(s))\,,$$

where $b(s) := \mathbf{E}[\, e^{-sB_1} \,]$ is the Laplace transform of the service time.

(a) Differentiate $h(s)$ once to obtain $\mathbf{E}[\, H_1 \,] = \frac{1}{\mu - \lambda}$. (We already saw this in class by noting that the busy and idle periods form an alternating renewal process, with the idle periods distributed as $\mathsf{Expon}(\lambda)$, hence $\frac{\mathbf{E}H_1}{\mathbf{E}H_1 + 1/\lambda} = \rho = \frac{\lambda}{\mu}$, by Little's law.)

(b) Specialize further to the case of **M/M/1 systems**. Differentiate $h(s)$ twice to obtain $\mathrm{Var}[H_1] = \frac{\lambda+\mu}{(\mu-\lambda)^3}$. Note that if λ is close to μ, then the standard deviation of H_1 (and the subsequent busy periods H_i) is huge, i.e., there are enormous fluctuations in the system.

Back to **M/G/1 systems**, we would also like to calculate the average limiting waiting time $\overline{W}$. The next exercise is a preparation for this.

▷ **Exercise 3.** For iid positive variables $B_1, B_2, \ldots$ with finite mean, consider $Z_n := B_1 + \cdots + B_n$. Take a random point $U_n \sim \mathsf{Unif}[0, Z_n]$, and let K_n be the index that satisfies $Z_{K_n-1} \le U_n < Z_{K_n}$. Show that B_{K_n} converges in distribution to the size-biased version $\widehat{B}$.

▷ **Exercise 4.** Notice that $W_n = R_n + \sum_{i \in \mathcal{Q}_n} B_i$, where R_n is the time remaining from servicing the current costumer (if there is one) at the time of the nth arrival, and $\mathcal{Q}_n$ is the queue at that moment.

- **(a)** * For an M/G/1 system, show using Exercise 3 that R_n converges in distribution to $\mathsf{Ber}(\rho) \cdot \mathsf{Unif}[0,1] \cdot \widehat{B}$, where ρ is the utilization ratio, $\widehat{B}$ is the size-biased service time, and the three factors are independent from each other. In particular, the expectation of the limit is $\frac{\rho}{2} \frac{\mathbf{E}[B^2]}{\mathbf{E}[B]}$.
- **(b)** Show by example that, without the Markovianity of the arrival process, the previous result is wrong in general.
- **(c)** From part (a) and Little's law, obtain the equation

$$\overline{W} = \frac{\rho}{2} \frac{\mathbf{E}[B^2]}{\mathbf{E}[B]} + \lambda \overline{W}\, \mathbf{E}[B],$$

then deduce the **Pollaczek-Khinchin formula**:

$$\overline{W} = \frac{\lambda\, \mathbf{E}[B^2/2]}{1 - \lambda\, \mathbf{E}[B]}.$$

(A different proof can be found in Durrett's EOSP Section 3.2.3.)

The last exercise is about first order homogeneous infinite buffer **fluid queuing models** (as in the Telek lecture notes), with an underlying irreducible finite state Continuous Time Markov Chain with infinitesimal generator $Q = (q_{i,j})_{i,j=1}^n$, stationary distribution $(\pi_i)_{i=1}^n$, and fluid change rates $(r_i)_{i=1}^n$. The diagonal matrix formed by these rates is denoted by R. Recall that any stationary density vector $p_i(x)_{i=1}^n$ of the fluid level satisfies the following system of ODE's: $p'(x)R = p(x)Q$.

▷ **Exercise 5.**

- **(a)** Show that a first order homogeneous infinite buffer fluid queuing model cannot be stable (i.e., the fluid level cannot have a stationary distribution) if $\sum_i \pi_i r_i > 0$.
- **(b)** Show by examples that, if $\sum_i \pi_i r_i < 0$, then the characteristic equation $\det(\lambda R - Q) = 0$ may or may not have a negative root $\lambda < 0$ with a non-zero vector $\phi \in \ker(\lambda R - Q)$ that has only non-negative entries. If we do have such a solution ϕ, then we get a stationary density of the form $p_i(x) = ce^{\lambda x}\phi(i)$.