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On p-stability in groups and fusion systems

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ABSTRACT

The aim of this paper is to generalise the notion of p-stability (p is an odd prime) in finite group theory to fusion systems. We first compare the different definitions of p-stability for groups and examine properties of p-stability concerning subgroups and factor groups. Motivated by Glauberman's theorem, we study the question of how $Qd(p)$ is involved in finite simple groups. We show that with a single exception a simple group involving $Qd(p)$ has a subgroup isomorphic to either $Qd(p)$ or a central extension of $Qd(p)$ by a cyclic group of order p. Then we define p-stability for fusion systems and characterise some of its properties. We prove a fusion theoretic version of Thompson's maximal subgroup theorem. We introduce the notion of section p-stability both for groups and fusion systems and prove a version of Glauberman's theorem to fusion systems. We also examine relationship between solubility and p-stability for fusion systems and determine the simple groups whose fusion systems are $Qd(p)$-free.

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Introduction

Throughout, let p be an odd prime. The concept of p-stability goes back to the middle of the 1960s. It was originally defined by D. Gorenstein and J.H. Walter in [14] but, since then, it has undergone several modifications. p-stability was investigated by G. Glauberman and also played a role in the classification of finite simple groups. In the 1960s, several different definitions of p-stability arose and, at a first sight, these definitions appear not to be equivalent. In Section 1 of the present paper we go around the notion of p-stability and examine some basic properties that do not seem to have been considered so far. We show that p-stability inherits to subgroups but not to factor groups. The smallest group which is not p-stable is the semidirect product of $SL_2(p)$ with an elementary Abelian group of order p^2 (acted on by $SL_2(p)$ in the natural way). Glauberman denoted this group by $Qd(p)$ and showed that a group does not involve $Qd(p)$ if and only if all of its sections are p-stable. For further investigation, we define the concept of section p-stability and give a new version of Glauberman's theorem (see 1.20).

Motivated by this result, we ask the question: Which finite simple groups involve $Qd(p)$? The obvious necessary condition for a group G to involve $Qd(p)$ is a Sylow p-subgroup of G to be non-Abelian. We discover that this is almost sufficient:

Theorem 1. *Let G be a finite simple group whose Sylow p-subgroups are non-Abelian. Then G involves $Qd(p)$ unless G is one of the groups (i) $PSU_3(q)$ with q a p-power; (ii) ${}^2G_2(q)$ with $q = 3^{2m+1}$ and $p = 3$; and (iii) $G_2(q)$ with $q^2 - 1 \equiv 3$ or $6 \pmod 9$ and $p = 3$; (iv) J_2 or J_3 with $p = 3$; (v) HS, McL, Co_2, Co_3 with $p = 5$; and (vi) J_4 with $p = 11$.*

Furthermore, we prove that if G involves $Qd(p)$, then G contains a subgroup which is a perhaps trivial central extension of $Qd(p)$ with the only exception $G = He$. The following theorem refines Glauberman's result for an arbitrary group to the case where G is simple.

Theorem 2. *Let G be a finite simple group. Then G is p-stable if and only if it does not involve $Qd(p)$. More precisely, G is p-stable if and only if has no subgroup isomorphic to $Qd(p)$ or a central extension of $Qd(p)$ by a cyclic group of order p, with exception $G = He$, the sporadic Held group. In this case, G contains an extension of $Qd(p)$ by a Klein 4-group.*

Our proof of Theorem 1 is divided into three parts: we examine the alternating groups and simple groups of Lie type in defining characteristic in Section 2. We investigate simple groups of Lie type in non-defining characteristic in Section 3. Finally, in Section 4, the sporadic simple groups are discussed.

Several properties of groups can be investigated 'locally', that is, within the normalisers of their non-trivial p-subgroups. Moreover, a group acts on its p-subgroups by conjugation. This action was extensively studied and led to the definition of a (saturated) fusion system. The concept was introduced by L. Puig in the 1990s and was originally called a 'Frobenius category' (see [24]). We give the precise definition of a fusion system in Section 5. For the last 2 decades, fusion systems have been studied extensively and many concepts of group theory (such as solubility or simplicity) were defined in the case of fusion systems. Also many group theoretical results have turned out to be true for fusion systems. Although p-stability has not been defined for fusion systems so far, $Qd(p)$-free fusion systems were examined in [20]. In Section 6 of the present paper we introduce the concept of p-stability for saturated fusion systems and investigate its basic properties. It turns out that there are some differences. For example, unlike the case of finite groups, solubility does not imply p-stability (not even for $p > 5$).

In Section 7, we show a fusion theoretic version of Thompson's maximal subgroup theorem (see [12, p. 295, Theorem 8.6.3]). This can be summarised in the following way:

Theorem 3. *Let $\mathcal{F}$ be a saturated fusion system defined on the p-group P. Let $\mathfrak{Q}$ be a collection of subgroups of P closed under $\mathcal{F}$-morphisms. Let $\mathfrak{N}$ be the set of normaliser systems of subgroups of P that are defined on elements of $\mathfrak{Q}$. Assume each element of $\mathfrak{N}$ is constrained and p-stable. Then $\mathfrak{N}$ has a unique maximal element.*

Then, in Section 8, we investigate $Qd(p)$-free fusion systems and show the following:

Theorem 4. *A group does not involve $Qd(p)$ if and only if its fusion system is $Qd(p)$-free.*

We define section p-stability for fusion systems and prove a fusion theoretic version of Glauberman's result (see Section 9):

Theorem 5. *A fusion system is section p-stable if and only if it is $Qd(p)$-free.*

As a consequence, we give a slight refinement of Glauberman's theorem, see Theorem 8.12.

As the Sylow p-subgroups of $Qd(p)$ are extraspecial of exponent p and order p^3, we study the fusion systems defined on this group in Section 10. We show that with trivial exceptions all of these fusion systems are non-p-stable and non-soluble.

Finally, we apply our group theoretic results to fusion systems and investigate the relationship between solubility, p-stability and section p-stability for fusion systems in Section 11.

1. Summary on p-stable groups

In the literature, we can find different definitions of p-stability for groups. The notion of p-stability appears first in [14, Definition 2, p. 171], then in [12, p. 268]. Later, Glauberman redefines this notion in [10, Definitions 2.1 and 2.3, p. 1104] and in [11, p. 22].

Unfortunately, the four definitions are (pairwise) different and it is not clear at all whether they are equivalent. For the sake of completeness, we cite all four definitions. Glauberman proves that the definition in [11] is equivalent to that in [12], but the one in a later edition of the same book (see [13]) appears to be non-equivalent to that in [12]. Later in the literature the definition in [11] is used (see e.g. in [16] or [26]). However, results from [10] have great importance and are oft cited, so the equivalence of these definitions might be crucial. In the following, we shall compare the two definitions by examining some properties of p-stability.

The original definition of Gorenstein and Walter is the following:

Definition 1.1 *(Gorenstein–Walter, 1964).* Let G be a finite group. Let S be the largest soluble normal subgroup of G. Let p be a prime that divides $|S|$. Let P be a Sylow p-subgroup of $O_{p',p}(S)$ and $Q \leqslant P$ such that (i) $O_{p'}(S)Q \lhd G$ and (ii) $O_p\big(N_G(Q)/C_G(Q)P\big) = 1$. We shall say that G is *p-stable* provided the following condition holds for any such subgroup Q:

> If A is a p-subgroup that normalises Q and satisfies the commutator identity $[Q, A, A] = 1$, then $A \subseteq PC_G(Q)$.

Gorenstein's advanced definition in [12]:

Definition 1.2 *(Gorenstein, 1968).* Let G be a finite group and p an odd prime. G is called *p-stable* if the following condition is satisfied:

> If K is a normal subgroup of G, P is a p-subgroup of K with $G = KN_G(P)$, and A is a p-subgroup of $N_G(P)$ such that $[P, A, A] = 1$, then
>
> $$AC_G(P)/C_G(P) \subseteq O_p\big(N_G(P)/C_G(P)\big).$$

In [13], the above group K is specified as $O_{p',p}(G)$.

The definition appearing in [10] is as follows:

Definition 1.3 *(Glauberman, 1968).* Let G be a finite group, let $p > 2$ be a prime, and let $\mathcal{M}(G)$ be the set of subgroups M of G maximal with respect to the property that $O_p(M) \neq 1$. G is said to be *p-stable* if for all $M \in \mathcal{M}(G)$ and for all p-subgroups Q of M such that $O_{p'}(M)Q \lhd M$, whenever an element $x \in N_M(Q)$ has the property that if

$$[Q, x, x] = 1,$$

then x maps into $O_p\big(N_M(Q)/C_M(Q)\big)$ under the natural homomorphism $N_M(Q) \to N_M(Q)/C_M(Q)$.

The revised definition of p-stability in [11] is the following:

Definition 1.4 *(Glauberman, 1971).* A group G is said to be *p-stable* if for all p-subgroups Q of G whenever an element $x \in N_G(Q)$ satisfies

$$[Q, x, x] = 1,$$

then x maps into $O_p\big(N_G(Q)/C_G(Q)\big)$ under the natural homomorphism $N_G(Q) \to N_G(Q)/C_G(Q)$.

Remark 1.5.

(i) It can be easily checked that Gorenstein's subgroups A can be substituted by single elements x. Moreover, let $x = x_p x_{p'} \in N_G(Q)$, where x_p and $x_{p'}$ are commuting p- and p'-elements, respectively. It is straightforward to check that if $[Q, x, x] = 1$, then $x_{p'} \in C_G(Q)$. As a consequence, it can be assumed that x is a p-element.

(ii) By any of the four definitions, every group with an Abelian Sylow p-subgroup is trivially p-stable.

(iii) If we set $K = G$ in Definition 1.2, we obtain Definition 1.4, so Gorenstein's definition implies Glauberman's one.

(iv) It is less obvious, what the connection between the complicated first definition and the other ones is. Since this definition was soon revisited by Gorenstein himself, we shall not discuss this connection here.

The smallest example for a group *not* being p-stable (by all four definitions but we only check Glauberman's definitions) is the group usually denoted by $Qd(p)$:

Example 1.6. The group $Qd(p)$ is defined as a semidirect product of a two-dimensional vector space V over $\mathbb{F}_p$ with the special linear group $SL_2(p)$ via the natural action:

$$Qd(p) = V \rtimes SL_2(p).$$

Clearly, $O_p(Qd(p)) = V \neq 1$, so $\mathcal{M}(G)$ consists solely of the group itself. Since $O_{p'}(Qd(p)) = 1$, the subgroup Q has to be normal in $Qd(p)$. Hence $Q = V$ (or 1, but this case is trivial). Now, V is self-centralising, so $N_{Qd(p)}(V)/C_{Qd(p)}(V) \cong SL_2(p)$. The element

$$x = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in SL_2(p)$$

satisfies the commutator relation $[Q, x, x] = 1$. Nevertheless, x is not contained in $O_p(SL_2(p))$ since the latter is trivial. In the literature, this group is of great importance.

Note that the Sylow 2-subgroups of $Qd(p)$ are isomorphic to those of $SL_2(p)$ and hence they are generalised quaternion groups.

The next lemma gives a well-known description of $Qd(p)$ as a matrix group (see Example 7.5 in [15, p. 494]):

Lemma 1.7. *$Qd(p)$ can be represented as a subgroup of $SL_3(p)$, namely, consisting of matrices of the form*

$$\begin{bmatrix} a & b & t \\ c & d & u \\ 0 & 0 & 1 \end{bmatrix},$$

where $ad - bc = 1$. This subgroup intersects $Z(SL_3(p))$ trivially and hence maps isomorphically into $PSL_3(q)$.

As already mentioned, we shall focus on the latter two definitions of Glauberman. The first question concerning p-stability is whether these two definitions are equivalent. This question is important especially as theorems proved with Definition 1.3 in [10] are often cited when using Definition 1.4 of p-stability. Nevertheless, this problem does not seem to have been dealt with.

A group G with $O_p(G) \neq 1$ which is p-stable according to Definition 1.4 also satisfies Definition 1.3, simply because more subgroups Q are considered there. There are also some natural questions concerning p-stability which do not seem to have been considered so far, such as whether a subgroup or a factor group of a p-stable group is necessarily p-stable (according to any of the definitions).

In the following, we answer the questions asked above. In [8, p. 82] it is shown that the semidirect product of A_8 with an elementary Abelian group of order 3^8 is 3-stable according to Definition 1.3 and it contains a subgroup isomorphic to $Qd(3)$. Hence this definition does not inherit to subgroups. However, we can prove the following proposition using Definition 1.4 of p-stability:

Proposition 1.8. *Let G be a group that is p-stable according to Definition 1.4. Let H be a subgroup of G. Then H is p-stable according to the same definition.*

Proof. Let Q be a p-subgroup of H. Set $C = C_G(Q)$, $N = N_G(Q)$, $\bar{N} = N/C$, $N_H = N_H(Q)$, $C_H = C_H(Q)$ and $\bar{N}_H = N_H/C_H$. As $C_H = C \cap N_H$, we have

$$\bar{N}_H \cong N_H C/C \leqslant \bar{N},$$

so the former can be naturally considered as a subgroup of the latter. Let $x \in N_H$ such that $[Q, x, x] = 1$. By Definition 1.4, $xC \in O_p(\bar{N}) \cap \bar{N}_H \subseteq O_p(\bar{N}_H)$, whence the lemma. $\square$

This proposition has three immediate consequences:

Corollary 1.9. *A group G satisfying Definition 1.4 also satisfies Definition 1.3.*

Proof. Assume G is p-stable according to Definition 1.4. Let $M \in \mathcal{M}(G)$ and let $Q \leqslant M$ with $QO_{p'}(M) \lhd M$. By Proposition 1.8 M is p-stable by Definition 1.4. Then for any $x \in N_M(Q)$ such that $[Q, x, x] = 1$ we have $xC_M(Q) \in O_p\big(N_M(Q)/C_M(Q)\big)$, proving G is p-stable according to Definition 1.3. □

Corollary 1.10. *Definition 1.3 does not imply Definition 1.4, hence the two definitions are* not *equivalent.*

Proof. By [8, p. 82], the group $G = V \rtimes A_8$ is 3-stable according to Definition 1.3, but it is certainly not p-stable according to Definition 1.4 as G contains a subgroup isomorphic to $Qd(3)$ which is not 3-stable. □

Corollary 1.11. *A group G is p-stable according to Definition 1.4 if and only if $N_G(Q)$ is p-stable for all non-cyclic p-subgroups Q of G.*

Proof. Note that $\mathrm{Aut}(Q)$ is Abelian if Q is cyclic. So cyclic p-subgroups of G satisfy the p-stability condition, and hence this only needs to be verified for non-Abelian subgroups Q. □

From now on, we use Definition 1.4 for p-stability (unless otherwise stated explicitly).

The next question is about factor groups. In [8, p. 88] it is shown that $G/O_{p'}(G)$ is p-stable if G is so. Although Gagen uses Definition 1.3, the proof can be easily carried over to Definition 1.4, too.

The next example shows that a factor group of a p-stable group need not be p-stable in general. We are thankful to professor O. Yakimova for pointing out this example.

Example 1.12. Let $p > 3$ and let X and Y be indeterminates over $\mathbb{F}_p$. Then the polynomial ring $\mathbb{F}_p[X, Y]$ can be viewed as an $\mathbb{F}_pSL_2(p)$-module via the action extending the natural operation on the 2-dimensional vector space $\langle X, Y\rangle_{\mathbb{F}_p}$. Let W be the $p+1$-dimensional subspace of $\mathbb{F}_p[X, Y]$ generated by the homogeneous polynomials of degree p. Then the elements X^p, $X^{p-1}Y, \ldots, XY^{p-1}$, Y^p form a basis of W and W is an $\mathbb{F}_pSL_2(p)$-submodule. W has a single submodule $V = \langle X^p, Y^p\rangle_{\mathbb{F}_p}$. Note that $SL_2(p)$ acts on V via its natural representation. Consider the group $G = W^* \rtimes SL_2(p)$, where W^* denotes the module contragredient to W. Since W^* has a factor module isomorphic to $V^* \cong V$, G has a factor group isomorphic to $Qd(p)$. However, it can be easily computed that the group G itself is p-stable.

In [10, Lemma 6.3.], Glauberman proved a characterisation of the groups all of whose sections are p-stable:

Theorem 1.13 *(Glauberman). Let G be a finite group. Then the following two conditions are equivalent:*

(i) All sections of G are p-stable;
(ii) G does not involve $Qd(p)$.

Theorem 1.13 implies that for $p \geqslant 5$ all p-soluble groups are p-stable. The converse is obviously false: there are plenty of simple groups whose Sylow p-subgroups are Abelian for some prime p.

Unfortunately, there is no nice characterisation of p-stable groups. It is not true that a non-p-stable group necessarily has a subgroup isomorphic to $Qd(p)$:

Example 1.14. The group $Qd(p)$ has a central extension with a cyclic group Z of order p: Let $E = \langle \tilde{a}, \tilde{b} \rangle$ be an extraspecial group of exponent p. Denote its centre by Z so that $Z = \langle [\tilde{a}, \tilde{b}] \rangle$. Then $E/Z \cong V$ (the normal subgroup of $Qd(p)$ of order p^2). Moreover, the images a and b under the homomorphism $E \to V$ of $\tilde{a}$ and $\tilde{b}$, respectively, generate V. It is well-known that the automorphism group of E has a subgroup isomorphic to $SL_2(p)$ and the action of $SL_2(p)$ on $\tilde{a}$ and $\tilde{b}$ is the same as on a and b. Let $\widetilde{Qd}(p) = E \rtimes SL_2(p)$ with the action just defined. Then $\widetilde{Qd}(p)$ is non-p-stable as it is proven by the subgroup $Q = E$ and $x \in SL_2(p)$ as in Example 1.6. It is easy to see that $\widetilde{Qd}(p)$ does not contain a subgroup isomorphic to $Qd(p)$.

As we shall see later, $\widetilde{Qd}(p)$ has a faithful representation as a subgroup of $GL_p(q)$ if $p|q-1$ (see Lemma 3.6.) In order to give some more examples of non-3-stable groups, we now construct $\widetilde{Qd}(3)$ as a subgroup of $GL_3(\mathbb{C})$.

Example 1.15. Let ϱ be a (complex) primitive third root of unity. We define the following complex matrices:

$$\tilde{a} = \begin{bmatrix} \varrho & 0 & 0 \\ 0 & \varrho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \tilde{b} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \varrho & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad t = \frac{1}{1-\varrho} \cdot \begin{bmatrix} 1 & 1 & 1 \\ \varrho & \varrho^2 & 1 \\ \varrho^2 & \varrho & 1 \end{bmatrix}.$$

A straightforward calculation shows that $E = \langle \tilde{a}, \tilde{b} \rangle$ is an extraspecial group of order 27 and exponent 3, whereas, $S = \langle x, t \rangle$ is isomorphic to $SL_2(3)$. Moreover, S normalises E and the operation of the elements x and t with respect to the basis a, b of $E/Z(E)$ is represented by the matrices $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, respectively. Therefore, $\langle \tilde{a}, \tilde{b}, x, t \rangle \cong \widetilde{Qd}(3)$.

The group in Example 1.15 can be modified to obtain two more non-3-stable groups of the same order:

Example 1.16. We keep the notation of Example 1.15. Let ϑ be a primitive ninth root of unity with $\vartheta^3 = \varrho$ and let $x^- = \vartheta^{-1}x$ and $x^+ = \vartheta x$. Define the groups $\widetilde{Qd}^-(3) = \langle a, b, x^-, t\rangle$ and $\widetilde{Qd}^+(3) = \langle a, b, x^+, t\rangle$. As the original group is 'twisted' by a scalar matrix, all three groups have the same image in $PSL_3(\mathbb{C})$ (namely, a subgroup isomorphic to $Qd(3)$). Hence all these groups are central extensions of $Qd(3)$ by a cyclic group of order 3. Moreover, the elements x^+ and x^- together with the subgroup E show that $\widetilde{Qd}^+(3)$ and $\widetilde{Qd}^-(3)$ are non-3-stable.

Remark 1.17. By construction, the group $\widetilde{Qd}^-(3)$ is contained in $SL_3(\mathbb{C})$ unlike the other two groups. An easy calculation shows that the centraliser of a Sylow 2-subgroup of $\widetilde{Qd}(3)$ (a subgroup of order 72) contains an elementary Abelian group of order 9, while that in any of the other two groups contains a cyclic group of order 9.

Further investigation shows that $\widetilde{Qd}^-(3)$ and $\widetilde{Qd}^+(3)$ have non-isomorphic Sylow 3-subgroups.

Moreover, the Sylow 3-subgroups of all three groups have exponent 9 and the those of $\widetilde{Qd}(3)$ and $\widetilde{Qd}^+(3)$ cannot be embedded into $(C_9 \times C_9) \rtimes C_3$, the largest subgroup of $SL_3(\mathbb{C})$ of exponent 9.

Let $q = \ell^s$ such that $3|q-1$. Then reduction modulo ℓ carries over the construction in Example 1.15 to $GL_3(q)$. To see this observe that $\mathbb{F}_q$ contains primitive third roots of unity in this case.

If, moreover, $9|q-1$, then $\mathbb{F}_q$ contains primitive ninth roots of unity as well, and hence the constructions of Example 1.16 are valid in $SL_3(q)$ and $GL_3(q)$.

Note that the above defined groups are minimal non-3-stable subject to containment. The question naturally arises: which groups are minimal non-p-stable? We do not answer this question in this paper, but in section 4, we shall see one more example for the prime $p = 3$.

Although Theorem 1.13 was proved with Definition 1.3 of p-stability, the result is often used with Definition 1.4. In fact, the theorem is cited in [11], where Definition 1.4 appears, without mentioning that the proof was worked out with another definition. However, the next result is clear by the above:

Proposition 1.18. *For a group G, the following are equivalent:*

(i) *All sections of G are p-stable according to Definition 1.3.*
(ii) *All sections of G are p-stable according to Definition 1.4.*

For the proof observe that if G has a non-p-stable section H/K according to Definition 1.4 proved by the subgroup $Q \leqslant H/K$ and the element $x \in N_{H/K}(Q)$, then the section $N_{H/K}(Q)$ of G is non-p-stable according to Definition 1.3 (proved by the same p-subgroup Q and element x).

After introducing some notation, we define a more general notion. For p-subgroups Q, R of G such that $R \lhd Q$, we let $N_G(Q/R)$ be the largest subgroup of G that acts by conjugation on Q/R and $C_G(Q/R)$ be the largest subgroup of $N_G(Q/R)$ that acts trivially on Q/R. Note that

$$N_G(Q/R) = N_G(Q) \cap N_G(R)$$

and

$$C_G(Q/R) = \{x \in N_G(Q/R) \mid [Q, x] \subseteq R\}.$$

Definition 1.19. A group G is said to be *section p-stable* if for all p-subgroups R and Q of G such that $R \lhd Q$, whenever an element $x \in N_G(Q/R)$ satisfies $[Q, x, x] \subseteq R$, then $xC_G(Q/R)$ is contained in $O_p\big(N_G(Q/R)/C_G(Q/R)\big)$.

Clearly, any section p-stable group is p-stable.

Proposition 1.20. *For a group G, the following are equivalent:*

(i) *G is section p-stable.*
(ii) *All sections of G are p-stable.*
(iii) *$N_G(R)/R$ is p-stable for all p-subgroups R of G.*

Proof. The equivalence of (i) and (iii) is clear by the isomorphism theorems. Also, the implication (ii) $\Rightarrow$ (iii) is trivial.

(i) $\Rightarrow$ (ii): Assume first that G is section p-stable and let H/K be a section of G. Let T be a p-subgroup of H/K. Denote by Q a Sylow p-subgroup of the preimage of T under the natural homomorphism $H \to H/K$. Let $R = Q \cap K$. Then $T = KQ/K \cong Q/R$. Assume an element $\bar{x} \in N_{H/K}(T)$ satisfies $[T, \bar{x}, \bar{x}] = 1$.

Let $x \in H$ be such that $xK = \bar{x}$. Observe that $Q^x \subseteq KQ$ as T is normalised by $\bar{x}$. Since Q is a Sylow p-subgroup of KQ, we have $Q^x = Q^k$ for some $k \in K$. Hence $xk^{-1} \in N_H(Q)$ is also a preimage of $\bar{x}$, so we may assume $x \in N_H(Q)$.

By assumption, $[Q, x, x] \subseteq K$, so $[Q, x, x] \subseteq Q \cap K = R$ as Q is normalised by x. Now, as G is section p-stable,

$$xC_G(Q/R) \in O_p\big(N_G(Q/R)/C_G(Q/R)\big) \cap \big(N_H(Q/R) \cdot C_G(Q/R)/C_G(Q/R)\big)$$

follows. Since

$$N_H(Q/R)/C_H(Q/R) \cong N_H(Q/R) \cdot C_G(Q/R)/C_G(Q/R),$$

the coset $xC_H(Q/R)$ is contained in a normal p-subgroup of the factor group $N_H(Q/R)/C_H(Q/R)$. The claim now follows because

$$N_H(Q/R)/C_H(Q/R) \cong N_{H/K}(T)/C_{H/K}(T).$$

(Observe that $N_H(KQ/K) = K \cdot N_H(Q/R)$ and $C_H(KQ/R) = K \cdot C_H(Q/R)$ hold by straightforward calculations.) □

By Theorem 1.13, a group is section p-stable if and only if it does not involve $Qd(p)$.

2. $Qd(p)$ as a section of simple groups

We now discuss the problem which simple groups involve $Qd(p)$. More specifically, we want to examine how the group $Qd(p)$ is involved in finite simple groups. This question is discussed in the next few sections. Besides this, we also determine whether the simple group in question is p-stable.

This section is devoted to alternating groups and simple groups of Lie type in defining characteristic.

Theorem 2.1. *The alternating group A_n has a subgroup which is isomorphic to $Qd(p)$ if and only if $n \geqslant p^2$. For $n < p^2$, $Qd(p)$ is not involved in A_n. Therefore, A_n is p-stable for $n < p^2$ and non-p-stable otherwise.*

Proof. As the Sylow p-subgroups of A_n are Abelian if $n < p^2$, $Qd(p)$ cannot be involved in A_n in this case.

$SL_2(p)$ has index p^2 in $Qd(p)$. The permutation representation of $Qd(p)$ on the (right) cosets of $SL_2(p)$ is faithful as $Qd(p)$ has no normal subgroup contained in $SL_2(p)$ rather than the trivial one. This permutation representation gives an embedding of $Qd(p)$ into A_{p^2} (observe that $Qd(p)$ has no subgroup of index 2) and hence into each A_n with $n \geqslant p^2$.

The statement on p-stability follows from the above. □

The description of $Qd(p)$ as in Lemma 1.7 gives the main part of the following theorem:

Theorem 2.2. *Let G be a simple group of Lie type of characteristic p. Then $Qd(p)$ is not involved in G if and only if G is of type A_1, ${}^2A_2$ or ${}^2G_2(3^{2n+1})$. If G is of type B_2 or ${}^2A_n$ with $n \geqslant 3$, then G has a subgroup isomorphic to $\widetilde{Qd}(p)$. In all other cases and also if G is of type ${}^2A_n$ with $n \geqslant 6$, G has a subgroup isomorphic to $Qd(p)$. Consequently, G is p-stable if and only if it does not involve $Qd(p)$.*

Proof. Note that the cases of ${}^2B_2$ and ${}^2F_4$ are irrelevant because they are defined in characteristic 2.

The Ree groups ${}^2G_2(3^{2n+1})$ have Abelian Sylow 2-subgroups, hence they cannot involve $Qd(p)$. The simple groups of type A_1 have Abelian Sylow p-subgroups, so they do not involve $Qd(p)$.

For the unitary groups $G = PSU_3(q) = {}^2A_2(q^2)$, we can use the description of a Sylow p-subgroup P of G as in [18, Satz 10.12, p. 242]. A straightforward calculation shows the following: If a conjugate of an element (different from 1) of P is contained in P, then the conjugating element lies in the normaliser $N_G(P)$. Now, $N_G(P)/P$ is cyclic and hence does not involve $Qd(p)$. Therefore, no p-local subgroup of G involves $Qd(p)$ and hence G does not involve it, either.

Let $G = Sp_4(q)$ and let $X \cong Sp_4(p)$ be a subgroup of G. It is well-known that the stabiliser in X of a non-zero vector of the natural $\mathbb{F}_pSp_4(p)$-module is isomorphic to $\widetilde{Qd}(p)$. As $|Z(G)| = 2$, $PSp_4(q)$ has a subgroup isomorphic to $\widetilde{Qd}(p)$.

Note that $SO_5(q)$ is isomorphic to $PSp_4(q)$.

For $n \geqslant 4$, the special unitary group $SU_n(q)$ contains a subgroup isomorphic to $Sp_4(q)$ and hence it has a subgroup isomorphic to $\widetilde{Qd}(p)$. Since $Z(SU_n(q))$ is a p'-group, the same is true for $PSU_n(q)$.

All the other simple groups of Lie type (A_n for $n \geqslant 2$, B_n, C_n for $n \geqslant 3$, D_n and ${}^2D_n$ for $n \geqslant 4$, E_n for $6 \leqslant n \leqslant 8$, F_4, G_2, ${}^2E_6$, and ${}^3D_4$) and also ${}^2A_n$ with $n \geqslant 6$ are known to have a subgroup isomorphic to a possibly trivial central factor of $SL_3(p)$ (for the exceptional groups, see also [22]). Thus they all have subgroups isomorphic to $Qd(p)$ by Lemma 1.7. □

3. The case of simple groups of Lie type in non-defining characteristic

In this section, we discuss the question how $Qd(p)$ is involved in simple groups of Lie type in non-defining characteristic. More precisely, G is a simple group of Lie type defined over the field $\mathbb{F}_q$, where q is a power of a prime $\ell \neq p$. This means p differs from the defining characteristic ℓ of G.

The main result of this section is the following:

Theorem 3.1. *Let G be a simple group of Lie type of characteristic $\ell \neq p$. Suppose that the Sylow p-subgroups of G are non-Abelian. Then one of the following holds:*

- (*i*) *G contains a subgroup isomorphic to $\widetilde{Qd}(p)$;*
- (*ii*) *Either $G \cong PSL_p(q)$ (with $p|q-1$) or $G \cong PSU_p(q)$ (with $p|q+1$) or $p = 3$, $G \cong {}^3D_4(q)$, $F_4(q)$, ${}^2F_4(q)$, (with $q = 2^{2m+1}$ $m > 0$), or ${}^2F_4(2)'$ and G contains a subgroup isomorphic to $Qd(p)$;*
- (*iii*) *$p = 3$, $9|q^2-1$, $G = G_2(q)$ and G contains a subgroup isomorphic to $\widetilde{Qd}^{-}(3)$;*
- (*iv*) *$p = 3$ and q^2-1 is not a multiple of 9, $G = G_2(q)$ and G has no section isomorphic to $Qd(3)$.*

Consequently, G is p-stable if and only if it is section p-stable.

The conditions on a prime p which guarantee that a Sylow p-subgroup of a simple group G is Abelian must be known to experts, but we have not found any reference. So we write down these in Proposition 3.2 for cases relevant to Theorem 3.1, that is, for the cases where G is a simple group of Lie type defined over the field $\mathbb{F}_q$, $q = \ell^s$ and $\ell \neq p$. Denote by $e_p(q)$ the order of q modulo p, that is, the smallest natural number i such that $p|q^i - 1$.

Proposition 3.2. *Let G be a simple group of Lie type in characteristic $\ell \neq p$.*

(1) *Suppose that $p = 3$ and the Sylow 3-subgroups of G are Abelian. Then one of the following holds:*
  - (i) $G \cong PSL_2(q)$, *where* $q > 2$;
  - (ii) $G \cong PSL_3(q)$, *where* $q - 1 \equiv 3$ *or* $6 \pmod 9$;
  - (iii) $G \cong PSL_n(q)$, *where* $3|q + 1$ *and* $2 < n < 6$;
  - (iv) $G \cong PSU_3(q)$, *where* $q > 2$ *and* $q + 1 \equiv 3$ *or* $6 \pmod 9$;
  - (v) $G \cong PSU_n(q)$, *where* $3|q - 1$ *and* $2 < n < 6$;
  - (vi) $G \cong B_2(q)$;
  - (vii) $G \cong {}^2B_2(q)$, *where* $q = 2^{2m+1}$ *and* $m > 0$;

(2) *Suppose that $p > 3$ and the Sylow p-subgroups of G are Abelian. Then one of the following holds:*
  - (i) $G \cong {}^2B_2(q)$, *where* $q = 2^{2m+1}$, $m > 0$;
  - (ii) $G \cong G_2(q)$;
  - (iii) $G \cong {}^2G_2(q)$, *where* $q = 3^{2m+1}$, $m > 0$;
  - (iv) $G \cong {}^2F_4(2)'$ *or* ${}^2F_4(q)$, *where* $q = 2^{2m+1}$, $m > 0$;
  - (v) $G \cong {}^3D_4(q)$;
  - (vi) $G \cong F_4(q)$;
  - (vii) $G \cong E_6(q)$, *where* $p > 5$ *or* $p = 5 \nmid q - 1$;
  - (viii) $G \cong {}^2E_6(q)$, *where* $p > 5$ *or* $p = 5 \nmid q + 1$;
  - (ix) $G \cong E_7(q)$, *where* $p > 7$ *or* $p = 5$ *or 7 and* $p \nmid q^2 - 1$;
  - (x) $G \cong E_8(q)$, *where* $p > 7$ *or* $p = 7 \nmid q^2 - 1$ *or* $p = 5$;
  - (xi) $G \cong PSL_n(q)$, *where* $n < e_p(q)p$;
  - (xii) $G \cong PSU_n(q)$, *where* $2 < n < 2e_p(q)p$ *if* $e_p(q)$ *is odd,* $2 < n < e_p(q)p$ *if* $e_p(q) \equiv 0 \pmod 4$ *and* $n < e_p(q)p/2$ *if* $e_p(q) \equiv 2 \pmod 4$;
  - (xiii) $G \cong B_n(q)$, *where q is odd and* $1 < n < e_p(q)p$ *if* $e_p(q)$ *is odd,* $1 < n < e_p(q)p/2$ *if* $e_p(q)$ *is even;*
  - (xiv) $G \cong C_n(q)$, *where* $2 < n < e_p(q)p$ *if* $e_p(q)$ *is odd,* $2 < n < e_p(q)p/2$ *if* $e_p(q)$ *is even;*
  - (xv) $G \cong D_n(q)$, *where* $3 < n < e_p(q)p$ *if* $e_p(q)$ *is odd and* $4 < n \leqslant e_p(q)p/2$ *if* $e_p(q)$ *is even;*
  - (xvi) $G \cong {}^2D_n(q)$, *where* $3 < n \leqslant e_p(q)p$ *if* $e_p(q)$ *is odd and* $4 < n < e_p(q)p/2$ *if* $e_p(q)$ *is even.*

The proof of these two results occupies the rest of the section.

Lemma 3.3. *Let m, n be positive integers, and let $c = \gcd(m, n)$, the greatest common divisor of m and n. Then $q^c - 1$ is the greatest common divisor of $q^m - 1, q^n - 1$. Furthermore, p divides $q^n - 1$ if and only if $e_p(q)$ divides n.*

Proof. The first statement is Hilfsatz 2(a) in [17]. The second is an elementary consequence of the first as $e_p(q)$ is the order of q in the multiplicative group $\mathbb{F}_p^*$ of the field of p elements. □

Linear and unitary groups

Lemma 3.4. *Let E be an extraspecial group of order p^3. If p divides $q-1$ (resp. $q+1$), then E is isomorphic to a subgroup of $GL_p(q)$ (resp., $GU_p(q)$).*

Proof. The statement on $GL_p(q)$ is well known. Let p divide $q+1$. Then E is isomorphic to a subgroup of $GL_p(q^2)$. As $p > 2$, a Sylow p-subgroup of $GU_p(q)$ is a Sylow p-subgroup in $GL_p(q^2)$, see [27, p. 532], whence the statement. □

Lemma 3.5. *Let $G = GU_n(q)$, and P a Sylow p-subgroup of G. If $e_p(q) \equiv 2 \pmod 4$, then P is a Sylow p-subgroup of $GL_n(q^2)$, otherwise P is isomorphic to a Sylow p-subgroup of $GL_l(q^2)$, where l is the integral part of $n/2$.*

Proof. If $e = e_p(q) \equiv 2 \pmod 4$, then this is in stated in [27, p. 532]. So we may assume that either $4|e$ or e is odd.

Note that $GU_n(q)$ contains a subgroup X isomorphic to $GL_l(q^2)$. It suffices to prove the result for $n = 2l + 1$. As

$$|GU_n(q)| = (q+1)(q^2-1)\cdots(q^n+1)q^a$$

and

$$|GL_l(q^2)| = (q^2-1)\cdots(q^{2l}-1)q^b$$

for some integers $b > a > 1$, the index of $GL_l(q^2)$ in $GU_n(q)$ equals

$$(q+1)(q^3+1)\cdots(q^n+1)q^{a-b}.$$

We show that this number is coprime to p. For this it suffices to observe that $q^i + 1$ is coprime to p for i odd. Suppose the contrary that $p|q^i + 1$ for some i. Then $p|q^{2i} - 1$. By Lemma 3.3, $e|2i$.

Let first e be odd. Then $e|i$ and hence $p|q^i - 1$, so $p \nmid q^i + 1$.

Now let $e = 2m$, where m is even. Then $m|i$ as $e|2i$. This is a contradiction as m is even, whereas, i is odd. □

The statement on the general linear group of Lemmas 3.6 and 3.11 can also be found in [1, (3B), p. 12]. For the sake of being self-contained and also because it is short, we present the entire proof.

Lemma 3.6. *Let $G = GL_p(q)$ (resp., $GU_p(q)$), so that $G' = SL_p(q)$ (resp., $SU_p(q)$). Suppose $p|q-1$ (resp. $p|q+1$). Then G contains a subgroup isomorphic to $\widetilde{Qd}(p)$. If $p > 3$, then this subgroup is contained in G'. Consequently, $Qd(p)$ is isomorphic to a subgroup of $PGL_p(q)$ (resp. $PGU_p(q)$) and is contained in $PSL_p(q)$ (resp., $PSU_p(q)$) if $p > 3$.*

Proof. Set $Z = Z(G)$. Let E be the extraspecial group of order p^3 and exponent p. By Lemma 3.4, there is a faithful representation φ: $E \to G$. Then the character χ of φ vanishes on $E \setminus Z(E)$ [5, 9.20]. Then $(\chi, \chi) = 1$, and hence φ is absolutely irreducible. (As q is coprime to $|E|$, the representation theory of E over $\overline{\mathbb{F}}_q$ is paralleled with that over the complex numbers.)

For $g \in SL_2(p) \leqslant \widetilde{Qd}(p)$, the characters of representations φ and φ^g coincide, so φ and φ^g are equivalent. Therefore, there is $h \in GL_p(\overline{\mathbb{F}}_q)$ such that $\varphi^g = \varphi^h$. As φ is absolutely irreducible, the $\mathbb{F}_q$-envelope of $\varphi(E)$ is $Mat_p(\mathbb{F}_q)$, and h induces an automorphism of $Mat_p(\mathbb{F}_q)$. By the Skolem–Noether theorem, h can be chosen in $G = GL_p(q)$. By Schur's lemma, h is unique up to a scalar multiple. So $g \mapsto h$ is a projective representation of $SL_2(q) \to G$. As the Schur multiplier of $PSL_2(p)$ is of order 2, every projective representation of $SL_2(q)$ arises from an ordinary one, so h can be chosen so that $g \mapsto h$ is an ordinary representation. If $p > 3$, then $\widetilde{Qd}(p)$ has no non-trivial Abelian quotient. Since G/G' is Abelian, it follows that G' contains a subgroup H isomorphic to $\widetilde{Qd}(p)$.

Let us now consider the case $G = GU_p(q)$. Assume first $p > 3$. By the previous paragraph, we can assume that $\widetilde{Qd}(p) \cong H \leqslant SL_p(q^2)$ and $E \leqslant G$. It is well known that there exists an involutive automorphism τ, say, of $GL_p(q^2)$ such that $GU_p(q)$ is exactly the fixed point subgroup of τ. Let $g \in H$, $x \in E$. Then $gxg^{-1} = \tau(gxg^{-1}) = \tau(g)x\tau(g)^{-1}$, whence $g^{-1}\tau(g)x(g^{-1}\tau(g))^{-1} = x$. As E is absolutely irreducible, by Schur's lemma, $g^{-1}\tau(g)$ is a scalar matrix, z_g, say, so $\tau(g) = z_g g$. One easily observes that the mapping $g \mapsto z_g$ is a homomorphism of $H \cong \widetilde{Qd}(p)$ into the group of scalar matrices of $GL_p(q^2)$. As $\widetilde{Qd}(p)$ is perfect for $p > 3$, we have $z_g = 1$, and hence $\tau(g) = g$, that is, $g \in SU_p(q)$.

The above argument has to be refined for $p = 3$. In this case, $GL_3(q^2)$ has a subgroup H isomorphic to $\widetilde{Qd}(3)$. Recall that a Sylow 3-subgroup of $GL_3(q^2)$ coincides with one of $U_3(q)$ and hence H can be assumed to have a Sylow 3-subgroup contained in $GU_3(q)$. The kernel of the mapping $g \mapsto z_g$ as in the previous paragraph contains both the derived subgroup H' and the Sylow 3-subgroup of H contained in $GU_3(q)$. As H is generated by these subgroups, $z_g = 1$ follows for all $g \in H$. Hence $H \leqslant GU_3(q)$.

Finally, let again $p > 3$. Observe that the centre of H is contained in $Z(G')$. Therefore, its image in $PSL_p(q)$ (resp. $PSU_p(q)$) is isomorphic to $Qd(p)$. □

Next we examine the case $p = 3$ not discussed completely in Lemma 3.6.

Lemma 3.7. *Let $p = 3$ and $G = PSL_3(q)$. Suppose that $3|q-1$.*

- (i) *If $q-1$ is not a multiple of 9, then the Sylow 3-subgroups of G are Abelian, and G has no section isomorphic to $Qd(3)$.*
- (ii) *If $q-1$ is a multiple of 9, then $Qd(3)$ is isomorphic to a subgroup of G. Moreover, $SL_3(q)$ has a subgroup isomorphic to $\widetilde{Qd}^{-}(3)$ but not one isomorphic to $\widetilde{Qd}(3)$ or $\widetilde{Qd}^{+}(3)$.*

Proof.

- (i) The order of G is $q^3(q-1)^2(q+1)(q^2+q+1)/3$. One easily observes that the 3-part of $|G|$ is 9, so the Sylow 3-subgroups of G are Abelian. Then $Qd(3)$ is not a section of G.
- (ii) Assume $9|q-1$. By Lemma 3.6, $GL_3(q)$ contains a subgroup X isomorphic to $\widetilde{Qd}(3)$ whose image in $PGL_3(q)$ is isomorphic to $Qd(3)$. Now, $X \cong E \rtimes (Q_8 \rtimes C_3)$, where Q_8 is a quaternion group. Moreover, $X' \cong E \rtimes Q_8$ is contained in $SL_3(q)$ and $X = X' \rtimes \langle x \rangle$, where $x^3 = 1$.
  Let 3^ϑ be the 3-part of $q-1$. Then a Sylow 3-subgroup P of $SL_3(q)$ is isomorphic to $(C_{3^\vartheta} \times C_{3^\vartheta}) \rtimes C_3$. A straightforward calculation shows that any subgroup of P of exponent 9 is contained in a subgroup isomorphic to $(C_9 \times C_9) \rtimes C_3$ obtained from P in the obvious way. However, this group does not contain a Sylow 3-subgroup of $\widetilde{Qd}(3)$ (see also Remark 1.17). Thus $x \notin SL_3(q)$ and hence $\det(x)^3 = 1 \neq \det(x)$. Let $\alpha \in \mathbb{F}_q$ such that $\alpha^3 = \det(x)$ and set $y = \alpha^{-1}x$. Let $Y = \langle X', y \rangle$. Then Y is contained in $SL_3(q)$ and the image of Y in $PGL_3(q)$ is equal to that of X whence the claim on G.
  Finally, Remark 1.17 implies that $SL_3(q)$ does not contain a subgroup isomorphic to $\widetilde{Qd}^{+}(3)$ and hence $Y \cong \widetilde{Qd}^{-}(3)$ whence the claim. □

Lemma 3.8. *Let $p = 3$ and $G = PSU_3(q)$. Suppose that $q+1$ is a multiple of 3. Then $Qd(3)$ is isomorphic to a subgroup of G if and only if $q+1$ is a multiple of 9. In this case, $SU_3(q)$ has a subgroup isomorphic to $\widetilde{Qd}^{-}(3)$ but not one isomorphic to $\widetilde{Qd}(3)$ or $\widetilde{Qd}^{+}(3)$. If $3 \nmid q+1$, then $Qd(3)$ is not a section of G.*

Proof. Suppose $9|q+1$. We have shown in the proof of Lemma 3.7 that $SL_3(q^2)$ contains a subgroup Y such that $Y/Z(Y) \cong Qd(3)$. Note that $|Z(SL_3(q^2))| = |Z(SU_3(q))| = 3$. Let τ be as in the proof of Lemma 3.6, so by the argument there $z_g := g^{-1}\tau(g) \in Z(SL_3(q^2))$, and hence $z_g \in SU_3(q)$. Then, applying τ to $\tau(g) = z_g g$, we have $g = \tau^2(g) = z_g\tau(g) = z_g^2 g$, whence $z_g^2 = 1$, $z_g = 1$. Therefore, $g = \tau(g)$ and hence $g \in SU_3(q)$. The statement on $\widetilde{Qd}^{\pm}(3)$ follows from Lemmas 3.7 and 3.5.

Conversely, let $G = PSU_3(q)$, where $q+1$ is not a multiple of 9. The order of G is $q^3(q+1)^2(q-1)(q^2-q+1)/3$. One easily observes that the 3-part of $|G|$ is 9, so the Sylow 3-subgroups of G are Abelian, whence the result. □

Lemma 3.9. *Let $n > p$ and $G = PSL_n(q)$ (resp., $PSU_n(q)$), where $p|q-1$ (resp. $p|q+1$). Then G contains a subgroup isomorphic to $\widetilde{Qd}(p)$.*

Proof. Consider the embedding $\nu : SL_p(q) \to SL_n(q), \quad x \mapsto \mathrm{diag}(x, \mathrm{Id}_{n-p})$. Then $\nu(SL_p(q)) \cap Z(SL_n(q)) = 1$. This provides an embedding $SL_p(q) \to PSL_n(q)$. So G has a subgroup isomorphic to $\widetilde{Qd}(p)$ for $p > 3$.

This can be refined to the case $p = 3$ by using the embedding

$$\mu : GL_3(q) \to SL_n(q), \quad x \mapsto \mathrm{diag}(x, \det x^{-1}, \mathrm{Id}_{n-4}).$$

If the matrix $\mathrm{diag}(x, \det x^{-1}, \mathrm{Id}_{n-4})$ is scalar, then either $x = \mathrm{Id}_n$ or $n = 4$ and $x = a \cdot \mathrm{Id}_3 \in GL_3(q)$. Moreover, in the latter case $\det x^{-1} = a^{-3} = a$ must hold, so $a^4 = 1$. As such, if $x \neq \mathrm{Id}$, then it is not contained in $\widetilde{Qd}(3) \leqslant GL_3(q)$. Therefore, the homomorphism $GL_3(q) \to PSL_4(q)$ is faithful when restricted to $\widetilde{Qd}(3)$, so G contains a subgroup isomorphic to $\widetilde{Qd}(3)$.

The proof for the case of unitary groups is similar. □

Lemma 3.10.

(*i*) *Let $g \in GL_n(q)$, $g^p = 1 \neq g$. Suppose that g is irreducible. Then $n = e_p(q)$.*
(*ii*) *Let $g \in GL_n(q)$, where $n = e_p(q)$. Then the Sylow p-subgroups of G are cyclic.*
(*iii*) *Let $2n = e_p(q)$. Then $GU_n(q)$ contains an element of order p if and only if n is odd.*

Proof.

(*i*) It follows from the formula for $|GL_n(q)|$ that $e := e_p(q) \leqslant n$, otherwise p does not divide the group order. As g is irreducible, the enveloping algebra $[g]$ of g is a field (by Schur's lemma). In addition, the natural $\mathbb{F}_q GL_n(q)$-module V is of shape $[g] \cdot v$ for some $v \in V$, so $\dim[g] \geqslant n$. In fact, $\dim[g] = n$ as the matrix algebra $\mathrm{Mat}_n(\mathbb{F}_q)$ is well known to contain no subfield of dimension greater than n over $\mathbb{F}_q$. It follows that $[g] \cong \mathbb{F}_{q^n}$, and hence p divides $q^n - 1$. By Lemma 3.3, e divides n. Then $\mathbb{F}_{q^n}$ contains a subfield F isomorphic to $\mathbb{F}_{q^e}$. As the multiplicative group of $\mathbb{F}_{q^n}$ is cyclic, we have $g \in F$, and hence $[g] \cong F$, which means $F \cong \mathbb{F}_{q^n}$, that is, $e = n$.
(*ii*) The assumption $n = e_p(q)$ is equivalent to saying that $\mathbb{F}_{q^n}$ contains an element of order p, whereas $\mathbb{F}_{q^i}$ for $i < n$ contains no such element. As $\mathbb{F}_{q^i}^*$ embeds into $GL_i(q)$, it follows that a subgroup of $GL_n(q)$ isomorphic to $\mathbb{F}_{q^n}^*$ contains a Sylow p-subgroup of $GL_n(q)$, which is cyclic.

(*iii*) Recall that

$$|GU_n(q)| = (q+1)(q^2-1)\cdot \ldots \cdot (q^n \pm 1)$$

according to whether n is even or odd. As $e_p(q) = 2n$, no term of the form $q^i - 1$ in the above formula is divisible by p. If some $q^i + 1$ is divisible by p, then so is $q^{2i} - 1$ and hence $2i = 2e_p(q)$ must hold. Then $i = n$ is an odd number and the claim is proved. □

Lemma 3.11. *Let $e = e_p(q)$.*

(*i*) *If $n \geqslant pe$, then $GL_n(q)$ has a subgroup isomorphic to $\widetilde{Qd}(p)$. If $n > 3$, then this subgroup is contained in $SL_n(q)$.*
(*ii*) *If $n \geqslant 2pe$, then $SU_n(q)$ has a subgroup isomorphic to $\widetilde{Qd}(p)$.*
(*iii*) *If e is even and $n \geqslant ep$, then $SU_n(q)$ has a subgroup isomorphic to $\widetilde{Qd}(p)$.*
(*iv*) *If $e \equiv 2 \pmod 4$ and $n \geqslant pe/2$, then $SU_n(q)$ has a subgroup isomorphic to $\widetilde{Qd}(p)$ except for the case $e = 2$, $p = 3$ and $n = 3$.*

Proof.

(*i*) Suppose first that $n = pe$. Set $Y = GL_p(q^e)$. By Lemma 3.6, Y contains a subgroup isomorphic to $\widetilde{Qd}(p)$. So it suffices to show that there is a homomorphism $Y \to GL_n(q)$ faithful on restriction to $\widetilde{Qd}(p)$. First, observe that, viewing $\mathbb{F}_{q^e}$ as a vector space of dimension e over $\mathbb{F}_q$, we obtain an embedding of $\mathbb{F}_{q^e}$ into $\mathrm{Mat}_e(\mathbb{F}_q)$, which yields an embedding of $\mathrm{Mat}_p(\mathbb{F}_{q^e})$ into $\mathrm{Mat}_{pe}(\mathbb{F}_q)$. Therefore, $Y = GL_p(q^e)$ embeds into $GL_{pe}(q)$.
Note that $n = 3$ if and only if $p = 3$, $e = 1$ and $n = ep$. If $p > 3$, then $\widetilde{Qd}(p)$ is perfect, so $\widetilde{Qd}(p)$ embeds into $SL_{pe}(q)$. If $p = 3$ and $e > 1$, then $p | q+1$, so $e = 2$. Let $\overline{Y}$ be the image of Y in $GL_6(q)$. Then the index of $\overline{Y} \cap SL_6(q)$ in $\overline{Y}$ divides $q-1$. So either $\widetilde{Qd}(3)$ embeds into $SL_6(q)$ or $\widetilde{Qd}(3)$ has a proper normal subgroup, whose index in $\widetilde{Qd}(3)$ divides $q-1$. So the index is coprime to 3, and hence is a 2-power as $|\widetilde{Qd}(3)| = 3^4 \cdot 8$. It is well known that $SL_2(3)$, and hence $\widetilde{Qd}(3)$, has no proper quotient group of 2-power order. It follows that $SL_6(q)$ has a subgroup isomorphic to $\widetilde{Qd}(3)$.
Finally, let $n > pe$. The case $p = 3$, $e = 1$ has already been handled in the proof of Lemma 3.9. Otherwise $SL_n(q)$ has a subgroup isomorphic to $SL_{pe}(q)$ and (*i*) follows from the above.

(*ii*) Suppose first that $(e,p) \neq (1,3)$. By part (*i*), $SL_{pe}(q)$ contains a subgroup isomorphic to $\widetilde{Qd}(p)$. There is an embedding $SL_{pe}(q) \to SU_{2pe}(q)$, whence the result.
Let $e = 1$, $p = 3$, so $3 | q-1$. Then $\widetilde{Qd}(3)$ is a subgroup of $GL_3(q)$ (see Lemma 3.6) and there is an embedding $GL_3(q) \to GU_6(q)$. Note that $GU_6(q)/SU_6(q)$ is of order $q+1$, which is coprime to 3. So either $\widetilde{Qd}(3)$ embeds into $SU_6(q)$ or $\widetilde{Qd}(3)$

has a proper normal subgroup, whose index in $\widetilde{Qd}(3)$ divides $q+1$. So the index is coprime to 3, and hence a 2-power as above. As $\widetilde{Qd}(3)$ has no proper quotient group of 2-power order, it follows that $SU_6(q)$ has a subgroup isomorphic to $\widetilde{Qd}(3)$.
Consequently, (ii) holds for $n = 2pe$ and hence for $n > 2pe$, too.

(iii) Let e be even and let $e' = e_p(q^2)$. Then $e' = e/2$. By part (i), $SL_{pe'}(q^2) = SL_{ep/2}(q^2)$ has a subgroup isomorphic to $\widetilde{Qd}(p)$ unless $n = 3$. As there is an embedding $SL_{pe/2}(q^2) \to SU_{ep}(q)$, the statement follows. For $n = 3$ we proceed as in part (ii).

(iv) Let $e = 2m$, where m is odd. Then p divides $q^m + 1$. By Lemma 3.6, $\widetilde{Qd}(p)$ is isomorphic to a subgroup of $SU_p(q^m)$, provided $p > 3$. By [17, Hilfsatz 1], there is an embedding $SU_p(q^m) \to SU_{pm}(q)$, whence the result follows for $p > 3$. If, however, $p = 3$ and hence $e = 2$, then $\widetilde{Qd}(3)$ is isomorphic to a subgroup of $GU_3(q)$ by Lemma 3.6. Since there is an embedding $GU_3(q) \to SU_n(q)$ for $n > 3$, the result follows. □

Next we show that if the assumptions of Lemma 3.11 fail, then the Sylow p-subgroups of G are Abelian.

Lemma 3.12. *Let $e = e_p(q)$.*

(i) *If $n < ep$, then the Sylow p-subgroups of $GL_n(q)$ and hence of $PSL_n(q)$ are Abelian.*

(ii) *If e is odd and $n < 2ep$, then the Sylow p-subgroups of $GU_n(q)$ and hence of $PSU_n(q)$ are Abelian.*

(iii) *If $e \equiv 0 \pmod 4$ and $n < ep$, then the Sylow p-subgroups of $GU_n(q)$ and hence of $PSU_n(q)$ are Abelian.*

(iv) *If $e \equiv 2 \pmod 4$ and $n < ep/2$, then the Sylow p-subgroups of $GU_n(q)$ and hence of $PSU_n(q)$ are Abelian.*

Proof.

(i) As $|GL_n(q)| = (q-1) \cdot \ldots \cdot (q^n-1)q^a$, the order of a Sylow p-subgroup of $GL_n(q)$ equals the p-part of $(q-1) \cdot \ldots \cdot (q^n-1)$. By Lemma 3.3, p divides $q^j - 1$ if and only if e divides j. Therefore, the p-part of $(q-1)\cdots(q^n-1)$ coincides with that of $(q^e-1)(q^{2e}-1) \cdot \ldots \cdot (q^{ke}-1)$ for some $k < p$.
We claim that p is coprime to $\frac{q^{ie}-1}{q^e-1}$ for $i < p$. Indeed,

$$\frac{q^{ie}-1}{q^e-1} = (q^{(i-1)e}-1) + \cdots + (q^e-1) + i,$$

whence the claim follows. Therefore, if p^d is the p-part of $q^e - 1$, then the p-part of $|GL_n(q)|$ equals p^{dk}, and coincides with that of $GL_k(q^e)$. In addition, p^{dk} coincides

with the p-part of the order of the group of diagonal matrices of $GL_k(q^e)$. Hence the latter is one of the Sylow p-subgroups of $GL_k(q^e)$ and these are Abelian.
Now, there is an embedding $GL_k(q^e) \to GL_n(q)$ and the p-parts of the orders of these groups are the same. So the Sylow p-subgroups of $GL_k(q^e)$ are isomorphic to those of $GL_n(q)$, whence the result.

(ii) By Lemma 3.5, the Sylow p-subgroups of $GU_n(q)$ are isomorphic to those of $GL_l(q^2)$, where l is the integral part of $n/2$. By assumption $n < 2ep$, so $l < ep$. Moreover, $e_p(q) = e_p(q^2)$ as this number is odd. Therefore, the Sylow p-subgroups of $GL_l(q^2)$ are Abelian by part (i) and the claim follows.

(iii) We proceed in a similar way as in part (ii). By Lemma 3.5, the Sylow p-subgroups of $GU_n(q)$ are isomorphic to those of $GL_l(q^2)$ with the same l. But now we have $l < ep/2$ and $e_p(q^2) = e_p(q)/2 = e/2$, so part (i) applies again and the Sylow p-subgroups under consideration are Abelian.

(iv) Now the Sylow p-subgroups of $GU_n(q)$ are isomorphic to those of $GL_n(q^2)$ and $e_p(q^2) = e/2$, so the assumption $n < ep/2$ ensures that part (i) can be applied and the result follows. □

Proposition 3.13.

(i) *Let $G = GL_n(q)$ or $GU_n(q)$. If the Sylow p-subgroups of G are non-Abelian, then G contains a subgroup isomorphic to $\widetilde{Qd}(p)$.*

(ii) *Let $G = PSL_n(q)$ or $PSU_n(q)$. If the Sylow p-subgroups of G are non-Abelian, then G contains a subgroup isomorphic to $\widetilde{Qd}(p)$ or $Qd(p)$.*

Proof.

(i) This follows from Lemmas 3.6, 3.11 and 3.12.

(ii) Suppose first that $p = n = 3$ and $3|q-1$ (resp., $3|q+1$). Then the Sylow 3-subgroups of G are Abelian if and only if $q-1$ (resp., $q+1$) is not a multiple of 9. So in this case the result follows from Lemmas 3.7 and 3.8 for $G = PSL_3(q)$ and $PSU_3(q)$, respectively.
Assume $p > 3$ or $n \neq 3$. If by Lemma 3.12 the Sylow p-subgroups of G are non-Abelian, then we are in one of the situations in Lemma 3.11 whence the result. □

Symplectic groups

Lemma 3.14. *Let $G = Sp_{2n}(q)$ and let P be a Sylow p-subgroup of G.*

(i) *If $e_p(q)$ is odd, then P is isomorphic to a Sylow p-subgroup of $GL_n(q)$.*

(ii) *If $e_p(q)$ is even, then P is a Sylow p-subgroup of $GL_{2n}(q)$. If, in addition, e divides $2n$, then a Sylow p-subgroup of G is contained in a subgroup isomorphic to $GU_{2n/e}(q^{e/2})$.*

Proof.

(i) Note that G contains a subgroup X isomorphic to $GL_n(q)$. Recall that

$$|Sp_{2n}(q)| = (q^2-1)(q^4-1)\cdot \ldots \cdot (q^{2n}-1)q^a$$

and

$$|GL_n(q)| = (q-1)(q^2-1)\cdot \ldots \cdot (q^n-1)q^b$$

for some integers $a > b > 0$. So the index of $GL_n(q)$ in $Sp_{2n}(q)$ is equal to $q^{a-b}(q+1)(q^2+1)\cdot \ldots \cdot (q^n+1)$. We show that the index is coprime to p. If $p|q^i+1$, then $p \mid q^{2i}-1$. Then, by Lemma 3.3, e divides $2i$ and hence i as e is odd. It follows that $p|q^i-1$, which is impossible since p is odd.

(ii) For the first statement, see [27, p. 531].
Let $e = e_p(q)$. To prove the second statement, we start by showing that G contains a subgroup isomorphic to $GU_l(q^m)$, where $m = e/2$ and $l = 2n/e$.
Observe first that $Sp_e(q)$ contains an element g, say, of order p since $p|q^e-1 \mid |Sp_e(q)|$. Then g is irreducible as an element of $GL_e(q)$ by the very definition of e. As $e|2n$, it follows that the natural $\mathbb{F}_qG$-module V is a direct sum of $2n/e$ non-degenerate subspaces of dimension e. One observes that there is a homogeneous element $h \in G$ of order p (in other words, $h = \mathrm{diag}(g, \ldots, g)$ under a suitable basis of V). Then $C_G(h) \cong GU_{2n/e}(q^m)$, see for instance [6, Lemma 6.6].
Furthermore, observe that $p|q^m+1$ as $p|q^{2m}-1 = (q^m-1)(q^m+1)$ and $p \nmid q^m-1$. Note that $p|q^{2i}-1$ implies $e|2i$, and hence $m|i$. Therefore, the p-part of $|G|$ divides

$$(q^e-1)(q^{2e}-1)\cdot \ldots \cdot (q^{2n}-1).$$

Consider the term

$$q^{ie}-1 = q^{2im}-1 = (q^{im}-1)(q^{im}+1)$$

with i odd. As $p|q^m+1$, and hence $p|q^{im}+1$, we observe that p is coprime to $q^{im}-1$. Similarly, if $i = 2j$ is even, then

$$q^{ie}-1 = q^{2je}-1 = (q^{je}-1)(q^{je}+1).$$

As p divides $q^{je}-1 = q^{im}-1$, it is coprime to $q^{je}+1$. Therefore, the p-part of $|G|$ divides

$$(q^m+1)(q^{2m}-1)(q^{3m}+1)(q^{me}-1)\cdot \ldots \cdot (q^{le} \pm 1)$$

according to whether l is odd or even.

Recall that

$$|GU_l(q^m)| = q^b(q^m+1)(q^{2m}-1)(q^{3m}+1)\cdot\ldots\cdot(q^{lm}-(-1)^l)$$

for some integer $b > 0$. Therefore, the p-part of $|G|$ is equal to that of $|GU_l(q^m)|$ and the lemma is proven. □

Proposition 3.15. *Let $G = Sp_{2n}(q)$ and set $e = e_p(q)$. The following are equivalent:*

(1) *G contains a subgroup isomorphic to $\widetilde{Qd}(p)$;*
(2) *a Sylow p-subgroup of G is non-Abelian;*
(3) *$n \geqslant ep$ if e is odd, and $2n \geqslant ep$ if e is even.*

Proof. By Lemma 3.14, the equivalence of (2) and (3) follows from a corresponding result for $GL_m(q)$ for $m = n$ or $2n$, see Lemmas 3.6, 3.11 and 3.12. The implication (1) ⇒ (2) is trivial. If e is odd, then (3) implies (1) by Lemma 3.11 as $GL_n(q)$ is a subgroup of G.

Let $e = 2m$ be even, so $p|q^m + 1$. Suppose first $2n = pe$. By part (*ii*) of Lemma 3.14, some Sylow p-subgroup of G is contained in a subgroup X isomorphic to $GU_p(q^m)$. As $p|q^m + 1$, by Lemma 3.6, X contains a subgroup isomorphic to $\widetilde{Qd}(p)$. If $2n > pe$, then G contains a subgroup isomorphic to $Sp_{pe}(q)$, so the result follows. □

Orthogonal groups

Lemma 3.16. *Let $G = O^-_{2n}(q)$ or $O_{2n+1}(q)$, $e = 2m$ and $2n = de$, where d is odd. Then a Sylow p-subgroup of G is contained in a subgroup X isomorphic to $GU_d(q^m)$.*

Proof. We first show that $G = O^-_{2n}(q)$ contains a subgroup isomorphic to $GU_d(q^m)$. Note that $O^-_e(q)$ contains an element g, say, of order p as $p|q^m+1$ which divides $|O^-_e(q)|$ by the order formula. Observe that g is irreducible as an element of $GL_e(q)$ by the very definition of e. As $e|2n$, it follows that V, the natural $\mathbb{F}_qG$-module, is a direct sum of $d = 2n/e$ non-degenerate subspaces of dimension e. As d is odd, these can be chosen of Witt index 1 (see [19, 2.5.11] and use Witt's theorem). One observes that there is a homogeneous element $h \in G$ of order p (under a suitable basis of V we have $h = \mathrm{diag}(g, \ldots, g)$). Then $C_G(h) \cong GU_d(q^m)$, see for instance [6, Lemma 6.6]. So $O^-_{2n}(q)$ and hence $O_{2n+1}(q)$ contains a subgroup X isomorphic to $GU_d(q^m)$.

So it suffices to show that the p-part of G does not exceed that of $GU_d(q^m)$, and in turn that the p-part of $O_{2n+1}(q)$ does not exceed that of $GU_d(q^m)$. However, $|SO_{2n+1}(q)| = |Sp_{2n}(q)|$, and the p-part of $|Sp_{2n}(q)|$ equals the p-part of $|GU_d(q^m)|$ by Lemma 3.14. So the result follows. □

Lemma 3.17. *Let $G = O_{2n+1}(q)$, q odd, and let P be a Sylow p-subgroup of G. If e is even, then P is isomorphic to a Sylow p-subgroup of $GL_{2n+1}(q)$. If e is odd, then P is isomorphic to a Sylow p-subgroup of $GL_n(q)$.*

Proof. For the first statement see [27, p. 532]. Let e be odd. Then $|G|/2$ coincides with $|Sp_{2n}(q)|$, and G contains a subgroup X isomorphic to $GL_n(q)$.

By Lemma 3.14, the order of a Sylow p-subgroup of $GL_n(q)$ coincides with that of $Sp_{2n}(q)$, and hence with $|P|$. So the result follows. $\square$

Proposition 3.18. *Let $G = O_{2n+1}(q)$ and $e = e_p(q)$. The following are equivalent:*

(i) G contains a subgroup isomorphic to $\widetilde{Qd}(p)$;
(ii) a Sylow p-subgroup of G is not Abelian;
(iii) $n \geqslant ep$ if e is odd, and $n \geqslant ep/2$ if e is even.

Proof. Note that if q is even, then $SO_{2n+1}(q) \cong Sp_{2n}(q)$ and the result follows from Proposition 3.15, so we can assume that q is odd.

By Lemma 3.17, the equivalence of (ii) and (iii) follows from a corresponding result for $GL_m(q)$ for $m = n$ or $2n$, see Lemma 3.12. The implication $(i) \Rightarrow (ii)$ is trivial. If e is odd, then (iii) implies (i) by Lemma 3.11 as G has a subgroup isomorphic to $GL_n(q)$.

Let $e = 2m$ be even. Then a Sylow p-subgroup of $O^-_{pe}(q)$ and of G is contained in a subgroup X isomorphic to $GU_p(q^m)$ (see Lemma 3.16). As $p|q^m+1$, by Lemma 3.11 (iv), X contains a subgroup isomorphic to $\widetilde{Qd}(p)$. If $2n \geqslant pe$, then G contains a subgroup isomorphic to $O^-_{pe}(q)$, so the result follows. $\square$

Lemma 3.19. *Let $G = O^{\pm}_{2n}(q)$, $n > 3$ and let P be a Sylow p-subgroup of G.*

(i) P is isomorphic to a Sylow p-subgroup of $O_{2n+1}(q)$ or of $O_{2n-1}(q)$.
(ii) If $p \nmid q^{2n} - 1$ (equivalently, $e \nmid 2n$), then P is isomorphic to a Sylow p-subgroup of both $O_{2n+1}(q)$ and $O_{2n-1}(q)$.
(iii) P remains a Sylow p-subgroup of $O_{2n+1}(q)$ if and only if either $e \nmid 2n$ or $e|n$ for $G = O^+_{2n}(q)$ and $e \nmid n$ for $G = O^-_{2n}(q)$.
(iv) If q is even, the above statements remain true if one replaces $O_{2i+1}(q)$ by $Sp_{2i}(q)$ for $i = n, n-1$.

Proof. Recall that p divides $q^i - 1$ if and only if e divides i (see Lemma 3.3).

For (i), see [27, p. 533] or observe that the statement easily follows from the formulas for the orders of these three groups. Recall that

$$|O^+_{2n}(q)| = 2q^{n(n-1)}(q^2-1)\cdot\ldots\cdot(q^{2(n-1)}-1)(q^n-1),$$
$$|O^-_{2n}(q)| = 2q^{n(n-1)}(q^2-1)\cdot\ldots\cdot(q^{2(n-1)}-1)(q^n+1)$$

and

$$|O_{2n+1}(q)| = 2q^{n^2}(q^2-1)\cdot\ldots\cdot(q^{2n}-1).$$

(ii) follows from that the orders of $O_{2n+1}(q)$ and $O_{2n-1}(q)$ differ in a factor $q^{2n-1}(q^{2n}-1)$.

For (iii) observe that P remains a Sylow p-subgroup of $O_{2n+1}(q)$ if and only if p does not divide the index $|O_{2n+1}(q) : G|$, which is $q^n + 1$ for $G = O^+_{2n}(q)$ and $q^n - 1$ for $G = O^-_{2n}(q)$. This happens if either $e \nmid 2n$ (so $p \nmid q^{2n} - 1$) or $e|n$ for $G = O^+_{2n}(q)$ and $e \nmid n$ for $G = O^-_{2n}(q)$.

Finally, (iv) follows from the fact that $SO_{2n+1}(q) \cong Sp_{2n}(q)$ for q even. □

Lemma 3.19 (iii) together with Propositions 3.15 (for q even) and 3.18 implies:

Proposition 3.20. *Let $G = O^{\pm}_{2n}(q)$. Then G contains no subgroup isomorphic to $\widetilde{Qd}(p)$ if and only if the Sylow p-subgroups of G are Abelian.*

Proof. It suffices to show that G contains $\widetilde{Qd}(p)$ if the Sylow p-subgroups of G are non-Abelian. By Proposition 3.18, this is true if the Sylow p-subgroups of $O_{2n-1}(q)$ are non-Abelian. Assume that this is not the case. Then, by Lemma 3.19(i), the Sylow p-subgroups of $O_{2n+1}(q)$ are non-Abelian, and Proposition 3.18 implies that $n = ep$ (for e odd) or $n = ep/2$ (for e even). By part (iii) of Lemma 3.19 we have $G = O^+_{2n}(q)$ if e is odd, and $G = O^-_{2n}(q)$ if e is even. In the former case G contains $GL_n(q) = GL_{ep}(q)$ which contains $\widetilde{Qd}(p)$ by Lemma 3.11. The latter case has been already dealt with in the proof of Proposition 3.18 (iii). □

Proposition 3.21.

(i) *Let $G = O^+_{2n}(q)$ and let P be a Sylow p-subgroup of G. If e is odd, then P is Abelian if and only if $n < ep$. If e is even, then P is Abelian if and only if $n - 1 < ep/2$.*

(ii) *Let $G = O^-_{2n}(q)$ and let P be a Sylow p-subgroup of G. If e is odd, then P is Abelian if and only if $n - 1 < ep$. If e is even, then P is Abelian if and only if $n < ep/2$.*

Proof. Suppose first that e is odd. By Proposition 3.18, the Sylow p-subgroups of G are Abelian if $n < ep$ (since those of $O_{2n+1}(q)$ are Abelian). Furthermore, the Sylow p-subgroups of G are non-Abelian if $n > ep$ (since those of $O_{2n-1}(q)$ are so). If, however, $n = ep$, then $e|n$ and hence by part (iii) of Lemma 3.19 the Sylow p-subgroups of $O^+_{2n}(q)$ are non-Abelian while those of $O^-_{2n}(q)$ are Abelian.

Let now e be even. By Proposition 3.18, the Sylow p-subgroups of G are Abelian if $n < ep/2$. Furthermore, the Sylow p-subgroups of G are non-Abelian if $n > ep/2$. If, however, $n = ep/2$, then $e|2n$ and $e \nmid n$, so by part (iii) of Lemma 3.19 the Sylow p-subgroups of $O^-_{2n}(q)$ are non-Abelian while those of $O^+_{2n}(q)$ are Abelian and the result follows. □

Exceptional groups of Lie type We first recall that for $p > 2$ the Sylow p-subgroups of the simple groups ${}^2B_2(q)$, $q > 2$ are Abelian and the group ${}^2B_2(2)$ is soluble. Therefore, these groups are not to be considered.

We use information provided in [9, p. 111]. For $p > 2$, a Sylow p-subgroup P of a simple group G of Lie type has an Abelian normal subgroup A and the order of the quotient group $P_W = P/A$ can be computed from the table in [9, p. 111]. In particular, if $P_W = 1$, then P is Abelian.

Write $|G| = q^a b$, where b is coprime to q. Let Φ_m be the m-th cyclotomic polynomial, that is, an (over the rationals) irreducible polynomial whose roots are precisely the primitive m-th roots of unity. Then Φ_m divides $x^m - 1$ but does not divide $x^i - 1$ for $i < m$. The table in [9, p. 111] provides the expressions of $b = b(G)$ in terms of the Φ_m's. For instance, for the twisted group ${}^2E_6(q)$, we have $b = \Phi_1^4\Phi_2^6\Phi_3^2\Phi_4^2\Phi_6^3\Phi_8\Phi_{10}\Phi_{12}\Phi_{18}$. Write each expression as $\prod_m \Phi_m^{r_m}$. Let m_0 be the least number m such that p divides $\Phi_m(q)$. In fact, $m_0 = e_p(q)$, but we prefer to keep here notation of [9]. In a given expression for b, let M be the set of numbers m of the form $m = p^k m_0$ for some integer $k > 0$ such that $r_m > 0$. Then $|P_W| = p^d$, where $d = \sum_{m\in M} r_m$. In particular, $P_W = 1$ if and only if M is empty (see [9, p. 111]).

We illustrate this with the example $G = {}^2E_6(q)$. If $p > 5$, then M is empty, so P is Abelian. If $m_0 = 1$ and $p = 5$, then again P is Abelian, but if $m_0 = 2$, then $|P_W| = 5$. (In this case P is non-Abelian but this is not explicitly mentioned in [9].)

We first consider the groups of type E. The analysis of the table in [9, p. 111] yields the following conclusion:

Lemma 3.22. *Let $G = E_6(q)$, $E_7(q)$, $E_8(q)$ or ${}^2E_6(q)$ and let P be a Sylow p-subgroup of G.*

(i) P is Abelian if $p > 7$ and non-Abelian if $p = 3$;
(ii) if $p = 7$, then P is Abelian unless $G = E_7(q)$ and $m_0 = 1$ or 2 or $G = E_8(q)$ and $m_0 = 1$ or 2;
(iii) if $p = 5$, then P is Abelian unless one of the following holds:
- *(a) $G = E_6(q)$, $m_0 = 1$;*
- *(b) $G = {}^2E_6(q)$, $m_0 = 2$;*
- *(c) $G = E_7(q)$, $m_0 = 1$ or 2;*
- *(d) $G = E_8(q)$, $m_0 = 1$, 2 or 4.*

Note that $m_0 \neq 6$ in case (d) as $m_0 = e_p(q) < p$.

We have to decide whether $\widetilde{Qd}(p)$ is a subgroup of G whenever the Sylow p-subgroups of G are non-Abelian. The following lemma is an extraction from [22, Table 5.1].

Lemma 3.23. *Let $G = E_6(q)$, $E_7(q)$, $E_8(q)$ or ${}^2E_6(q)$. Suppose that the Sylow p-subgroups of G are non-Abelian. Then $\widetilde{Qd}(p)$ is a subgroup of G.*

Proof. We use information from [22, Table 5.1].

Suppose first that $G \cong E_6(q)$ (resp., ${}^2E_6(q)$). Then two primes: $p = 3$ and $p = 5$ have to be considered. Set $X = SL_6(q)$ (resp., $X = SU_6(q)$) and $X_1 = SL_5(q)$ (resp., $X_1 = SU_5(q)$). By [22, Table 5.1], G contains a subgroup isomorphic to X/Z, where Z is a central subgroup of X. Let first $p = 3$. Then by Lemma 3.11, X and $X/Z(X)$ and hence also X/Z contain a subgroup isomorphic to $\widetilde{Qd}(3)$. Let now $p = 5$, so $m_0 = 1$ (resp., 2). The natural embedding $X_1 \to X$ yields an embedding $X_1 \to X/Z$. By Lemma 3.6 X_1 contains a subgroup isomorphic to $\widetilde{Qd}(5)$ whence the result.

Suppose now that $G = E_7(q)$. Then $p = 3$, 5 and 7 have to be considered. By [22, Table 5.1], G has a subgroup X isomorphic to $\Omega^+_{12}(q)$. We use Propositions 3.20 and 3.21. Since $n = 6 > 3 = 1 \cdot 3 = 2 \cdot 3/2$ and $6 > 5$, X contains subgroups isomorphic to $\widetilde{Qd}(3)$ and $\widetilde{Qd}(5)$. Let now $p = 7$, so $m_0 = 1$ or 2. By [22, Table 5.1], G contains subgroups isomorphic to a central quotient of $SL_8(q)$ and of $SU_8(q)$. Therefore, G contains subgroups isomorphic to $SL_7(q)$ and $SU_7(q)$. So the result follows from Lemma 3.6.

Finally, let $G = E_8(q)$. Then G has a subgroup isomorphic to $\Omega^+_{16}(q)$, so we have $n = 8$ in Propositions 3.20 and 3.21. Then ep or $ep/2$ in question are 3 (for $p = 3$), 5, 5 and 10 (for $p = 5$) and 7 (for $p = 7$). Since only 10 exceeds 8, we are left with the case $p = 5$ and $m_0 = 4$. Again by Table 5.1 in [22], G has a subgroup isomorphic to $SU_5(q^2)$. As $m_0 = 4$, $p|q^2 + 1$. So $SU_5(q^2)$, and hence G, has a subgroup isomorphic to $\widetilde{Qd}(5)$. This completes the proof. □

Using [9, p. 111], we conclude that for $p > 3$, the Sylow p-subgroups of the groups ${}^3D_4(q)$, $F_4(q)$, ${}^2F_4(q)$ ($q = 2^{2m+1}$), ${}^2F_4(2)'$, $G_2(q)$, ${}^2G_2(q)$, ($q = 3^{2m+1}$) are Abelian. As we assume that q is not a p-power, the groups ${}^2G_2(q)$ for $p = 3$ are not to be considered here.

Lemma 3.24. *Let $p = 3$, $3 \nmid q$.*

(*i*) *If $G = {}^3D_4(q)$, $F_4(q)$, ${}^2F_4(q)$ (with $q = 2^{2m+1}$, $m > 0$) or ${}^2F_4(2)'$, then G contains a subgroup isomorphic to $Qd(3)$.*

(*ii*) *If $G = G_2(q)$ and $9 \nmid q^2 - 1$, then the Sylow 3-subgroups of G are non-Abelian and G contains no section isomorphic to $Qd(3)$.*

(*iii*) *If $G = G_2(q)$ and $9|q^2 - 1$, then G contains a subgroup isomorphic to $\widetilde{Qd}^-(3)$.*

Proof. Let $G = {}^3D_4(q)$. By [21, p. 182], G contains a subgroup X isomorphic to $PGL_3(q)$ (resp., $PGU_3(q)$) if $3|q - 1$ (resp., $3|q + 1$). By Lemma 3.6 X has a subgroup isomorphic to $Qd(3)$ whence the claim.

Let $G = F_4(q)$. Then G contains a subgroup isomorphic to ${}^3D_4(q)$ (see [22, Table 5.1]), so the result follows from that for ${}^3D_4(q)$.

Let $G = {}^2F_4(2)'$. Then G contains a subgroup isomorphic to $PSL_3(3)$ by [3]. By Lemma 1.7, the latter and hence G has a subgroup isomorphic to $Qd(3)$.

Let now $G = {}^2F_4(q)$, $q = 2^{2m+1} > 2$, $m > 0$. By Lemma 2.2(6) in [23], G contains a subgroup isomorphic to ${}^2F_4(2)'$, so the result follows from the previous paragraph.

Let $G = G_2(q)$. Then there are two maximal subgroups D_1, D_2 of G with non-Abelian Sylow 3-subgroups; moreover, D_1 contains $SL_3(q)$, D_2 contains $SU_3(q)$ as a subgroup of index 2 (see [22, Table 5.1]). If $9|q - 1$ (resp., $9|q + 1$), then $SL_3(q)$ (resp., $SU_3(q)$) has a subgroup isomorphic to $\widetilde{Qd}^-(3)$ by Lemmas 3.7 and 3.8. If, however, $9 \nmid q^2 - 1$, then a Sylow 3-subgroup E of G is extraspecial of order 27 and exponent 3. Therefore, if $Qd(3)$ is involved in G, then it must be involved either in the normaliser of E or in the normaliser of some elementary Abelian subgroup V of E. Let $Z = Z(E)$. Then $N_G(E) \subseteq N_G(Z)$, which has a subgroup of index 2 isomorphic to either $SL_3(q)$ or $SU_3(q)$ according to whether $3|q-1$ or $3|q+1$ (see [7, p. 461]). By Lemmas 3.7 and 3.8, these groups do not involve $Qd(3)$. Let us consider the other case. As V is normal in E, it must contain Z. Now, all elements of $E \setminus Z$ are conjugate in $C_G(Z)$ and they are not conjugate to an element of Z in G (see [7, p. 461]). Thus $N_G(V) \subseteq N_G(Z)$, which has been proved not to involve $Qd(3)$ whence the claim. □

Thus, we can sum the above arguments to get

Proposition 3.25. *Let G be a simple group of exceptional Lie type. Suppose that a Sylow p-subgroup of G is not Abelian. If $p > 3$, then $\widetilde{Qd}(p)$ is a subgroup of G.*

If $p = 3$, this is true if $G \cong E_6(q)$, $E_7(q)$, $E_8(q)$ or ${}^2E_6(q)$. Otherwise G contains a subgroup isomorphic to $Qd(3)$ unless $G \cong G_2(q)$. In the latter case G contains a subgroup isomorphic to $\widetilde{Qd}^-(3)$ if $9|q^2-1$ and has no section isomorphic to $Qd(3)$ if $9 \nmid q^2-1$.

4. The case of the sporadic groups

Having a look at the orders of the sporadic groups, we find only few primes to consider as a group having a $Qd(p)$-section must have a Sylow p-subgroup of order at least p^3. The primes together with the relevant groups are the following:

- For $p = 3$: M_{12}, M_{24}, J_2, J_3, J_4, Co_1, Co_2, Co_3, Fi_{22}, Fi_{23}, Fi_{24}', McL, He, Ru, Sz, $O'N$, HN, Ly, Th, B, M.
- For $p = 5$: Co_1, Co_2, Co_3, HS, McL, Ru, HN, Ly, Th, B, M.
- For $p = 7$: Fi_{24}', He, $O'N$, M.
- For $p = 11$: J_4.
- For $p = 13$: M.

A non-trivial section of a simple group is a section of one of its maximal subgroups. In the following examination we use the results listed in the Atlas of finite simple groups, see [3], or [28]. Since we employ results of the Atlas, it seems to be reasonable to keep Atlas notation in this section.

- $p = 3$:
  The maximal subgroups of J_2 with order divisible by 27 are $U_3(3)$ and $3.A_6.2$. As none of them involves $Qd(3)$, J_2 does not either. Similarly, the only maximal subgroups of J_3 with order divisible by 27 are $(3 \times A_6) : 2_2$ and $3^{2+1+2} : 8$. As none of them involves $Qd(3)$, J_3 is section 3-stable.

M_{12} has a maximal subgroup of type $3^2{:}2S_4$. Note that 3^2 is self-centralising and $2S_4 = GL_2(3)$ here. Hence this maximal subgroup contains a subgroup isomorphic to $Qd(3)$. Therefore, the simple groups M_{12}, M_{24}, J_4, Co_1, Co_3, Fi_{22}, Fi_{23}, Fi'_{24}, Sz, HN, B, M all contain subgroups isomorphic to $Qd(3)$ and hence they are non-3-stable.
McL contains a maximal subgroup of type $U_4(3)$. By Theorem 2.2, $U_4(3)$ has a subgroup isomorphic to $Qd(3)$. Hence each of the groups McL, Co_2 and Ly contains a subgroup isomorphic to $Qd(3)$, as they are overgroups of McL. Consequently, all these groups are non-3-stable.
The derived subgroup of the normaliser of $3A^2$ in He has structure $(2^2 \times 3^2){\cdot}SL_2(3)$. This is a non-split extension $2^2{\cdot}Qd(3) = 3^2{:}(2^2{\cdot}SL_2(3))$. This group is a new example for a minimal non-3-stable group.
Ru has a maximal subgroup of type ${}^2F_4(2)'.2$. By Proposition 3.2 the Sylow 3-subgroups of the latter are non-Abelian. Thus by Theorem 3.1 ${}^2F_4(2)'.2$ and hence Ru contains a subgroup isomorphic to $Qd(3)$. As a consequence, Ru is non-3-stable.
The Sylow 3-subgroups of $O'N$ are elementary Abelian. Hence $O'N$ has no section isomorphic to $Qd(3)$ and hence it is section 3-stable.
Th has a maximal subgroup of type $U_3(8){:}6$. By Proposition 3.2, the Sylow 3-subgroups of $U_3(8)$ are non-Abelian. Thus by Theorem 3.1, $U_3(8)$ and hence Th contains a subgroup isomorphic to $Qd(3)$. Therefore, Th is non-3-stable.

- $p = 5$:
The only maximal subgroup of HS with order divisible by 125 is $U_3(5) : 2$. By Theorem 2.2, this group and hence HS have no section isomorphic to $Qd(5)$. Thus it is section 5-stable.
The only non-soluble maximal subgroup of McL with the required order is $U_3(5)$, so McL has no section isomorphic to $Qd(5)$ whence it is section 5-stable.
The maximal subgroups of Co_2 with adequate order are McL and $HS : 2$. Those for Co_3 are $McL.2$, HS, and $U_3(5) : S_3$. Hence none of these groups has a section isomorphic to $Qd(5)$, so they are all section 5-stable.
Co_1 has a maximal subgroup $5^2{:}2A_5$ which is nothing else but $Qd(5)$. We remark that Co_1 has a maximal subgroup $5^{1+2}{:}GL_2(5)$, which has a subgroup isomorphic to $\widetilde{Qd}(5)$. As a consequence, Co_1 is non-5-stable.
Ru has a maximal subgroup of type $5^2{:}4S_5$, which contains a subgroup isomorphic to $Qd(5)$ and hence Ru is non-5-stable.
Th has a maximal subgroup of type $5^2{:}GL_2(5)$. Therefore, Th and its overgroups, B and M have subgroups isomorphic to $Qd(5)$. Thus they are non-5-stable.
HN has a maximal subgroup of type $5^2.5^{1+2}_{+} : 4A_5$. Here, $4A_5$ contains $SL_2(5)$, which operates on 5^2 on the natural way. Hence HN has a subgroup isomorphic to $Qd(5)$, so it is not 5-stable.
Ly has a maximal subgroup of type $G_2(5)$, which has a subgroup isomorphic to $Qd(5)$ by Theorem 2.2. Therefore, Ly is non-5-stable.

- $p = 7$:
  He has a maximal subgroup of type $7^2{:}2.L_2(7)$, which is isomorphic to $Qd(7)$. Hence He, Fi'_{24} and M all have subgroups isomorphic to $Qd(7)$ and they are not 7-stable. The group $O'N$ has a maximal subgroup of type $L_3(7){:}2$. Hence by Lemma 1.7, it also has a subgroup isomorphic to $Qd(7)$ and is therefore non-7-stable.
- $p = 11$:
  J_4 has two maximal subgroups of order divisible by 11^3. These are $U_3(11){:}2$ and $11^{1+2}{:}(5 \times 2S_4)$. None of them has a section isomorphic to $Qd(11)$, so J_4 has no one either. Therefore, L_4 is section 7-stable.
- $p = 13$:
  We find that the monster group M has a maximal subgroup with structure $13^2{:}2L_2(13).4$, so $Qd(13)$ is a subgroup of M and hence it is not 13-stable.

We summarise the above considerations as follows:

Theorem 4.1. *Let G be a sporadic simple group. Then G is p-stable if and only if it is section p stable. Otherwise, either $G = He$, $p = 3$ and G contains a subgroup of type $3^2{:}(2^{2\cdot}SL_2(3))$ or G contains a subgroup isomorphic to $Qd(p)$ and one of the following holds:*

(i) $G = M_{12}, M_{24}, J_4, Co_1, Co_2, Co_3, Fi_{22}, Fi_{23}, Fi'_{24}, McL, Ru, Sz, HN, Ly, Th, B$ *or* M *and* $p = 3$*;*
(ii) $G = Co_1, Ru, HN, Ly, Th, B$ *or* M *and* $p = 5$*;*
(iii) $G = Fi'_{24}, He, O'N$ *or* M *and* $p = 7$*;*
(iv) $G = M$ *and* $p = 13$*.*

5. Summary on fusion systems

In this section we recall the basic facts on fusion systems especially those we need later. First of all, we give the definition of a saturated fusion system following [20]. All fusion systems we deal with are saturated, so we shall omit the word 'saturated' in the sequel.

Let p be a prime and let P be a finite p-group. A fusion system $\mathcal{F}$ on P is a category whose objects are the subgroups of P and whose morphisms are certain injective group homomorphisms which will be written from the right.

The main example of a fusion system is that of a finite group G with Sylow p-subgroup P. If Q and R are subgroups of P such that $Q^g \leqslant R$ for some element $g \in G$ (that is, Q is subconjugate to R), then conjugation with g gives rise to a map $c_{g,Q,R}$: $Q \to R$ defined by $x \mapsto g^{-1}xg$ for $x \in Q$. The morphisms in the fusion system $\mathcal{F}_P(G)$ of G on P are exactly these maps so that

$$\mathrm{Hom}_{\mathcal{F}_P(G)}(Q, R) = \{c_{g,Q,R} \mid g \in G \text{ s.t. } Q^g \leqslant R\}.$$

The definition of an abstract fusion system $\mathcal{F}$ extracts the properties of $\mathcal{F}_P(G)$. To give the exact definition, we need some more notions.

- A subgroup Q of P is called fully $\mathcal{F}$-normalised if $|N_P(Q)| \geqslant |N_P(Q\varphi)|$ for every morphism $\varphi \in \mathcal{F}$ with domain Q.
- For an isomorphism φ: $Q \to R$ we let

$$N_\varphi = \{a \in N_P(Q) \mid \text{ there is } b \in N_P(R) \text{ such that } (a^{-1}xa)\varphi = b^{-1}(x\varphi)b \\ \text{for all } x \in R\}.$$

This means that the following diagram commutes:

$$\begin{array}{ccc} Q & \xrightarrow{\varphi} & R \\ {\scriptstyle c_{a,Q,Q}}\downarrow & & \downarrow{\scriptstyle c_{b,R,R}} \\ Q & \xrightarrow[\varphi]{} & R \end{array}$$

Note that if φ can be extended to a subgroup H of $N_P(Q)$, then $H \leqslant N_\varphi$.

Definition 5.1 *(Fusion system).* A *fusion system on the p-group P* is a category $\mathcal{F}$ with the subgroups of P as objects. Morphisms are injective group homomorphisms with the usual composition of functions such that the following hold:

(*i*) For all Q, $R \leqslant P$ the set $\mathrm{Hom}_P(Q, R)$ consisting of the P-conjugations from Q into R is contained in $\mathrm{Hom}_\mathcal{F}(Q, R)$.
(*ii*) For all morphisms $\varphi \in \mathrm{Hom}_\mathcal{F}(Q, R)$, the isomorphism $\bar{\varphi}$: $Q \to Q\varphi$ with $x \mapsto x\varphi$ (for all $x \in Q$) and $\bar{\varphi}^{-1}$: $Q\varphi \to Q$, $x\varphi \to x$ are morphisms in $\mathcal{F}$.
(*iii*) $\mathrm{Aut}_P(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_\mathcal{F}(P)$.
(*iv*) If Q is fully $\mathcal{F}$-normalised, then each $\mathcal{F}$-isomorphism φ: $R \to Q$ (where $R \leqslant P$) extends to an $\mathcal{F}$-morphism $\tilde{\varphi}$: $N_\varphi \to P$.

We now collect some notions concerning fusion systems that we shall use in this paper.

- A subgroup Q of P is called *strongly $\mathcal{F}$-closed* if for all subgroups R of Q and for all morphisms φ with domain R, the image $R\varphi$ is contained in Q.
- The *normaliser* of a fully $\mathcal{F}$-normalised subgroup Q of P is the subsystem $\mathcal{N}_\mathcal{F}(Q)$ of $\mathcal{F}$ defined on $N_P(Q)$ such that for $R, T \leqslant N_P(Q)$ the morphism $\varphi \in \mathrm{Hom}_\mathcal{F}(R, T)$ is in $\mathrm{Hom}_{\mathcal{N}_\mathcal{F}(Q)}(R, T)$ if φ extends to a morphism $\tilde{\varphi}$: $RQ \to TQ$ such that the restriction $\tilde{\varphi}_Q$ is an $\mathcal{F}$-automorphism of Q.

- Q is *normal* in $\mathcal{F}$, denoted by $Q \lhd \mathcal{F}$, if $\mathcal{F} = \mathcal{N}_{\mathcal{F}}(Q)$.
- If Q is normal in $\mathcal{F}$, a *quotient fusion system* $\mathcal{F}/Q$ can be defined on P/Q with morphisms $\bar{\varphi}$: $T/Q \to R/Q$ induced by morphisms φ: $T \to R$.
- $\mathcal{F}$ is called *soluble* if there is a sequence

$$1 = Q_0 < Q_1 < Q_2 < \ldots < Q_r = P$$

  with $Q_i/Q_{i-1} \lhd \mathcal{F}/Q_{i-1}$ for all $1 \leqslant i \leqslant r$.
- $O_p(\mathcal{F})$ is the largest normal subgroup of P that is normal in $\mathcal{F}$.
- A subgroup Q of P is called $\mathcal{F}$*-centric* if $C_P(Q\varphi)$ is contained in $Q\varphi$ for all morphisms φ with domain Q.
- $\mathcal{F}$ is said to be *constrained* if $C_P(O_p(\mathcal{F})) \subseteq O_p(\mathcal{F})$.
- A *model* of a constrained fusion system $\mathcal{F}$ is a p-constrained and p'-reduced group L (i.e. $C_L(O_p(L)) \subseteq O_p(L)$ and $O_{p'}(L) = 1$) with Sylow p-subgroup P such that $\mathcal{F} = \mathcal{F}_P(L)$. Note that each constrained fusion system has a model which is unique up to isomorphism, see [2, Proposition C].

6. Definition of p-stability for fusion systems

In this section, we define p-stable fusion systems and investigate their properties.

Observe first that the commutator of two group elements can be written in terms of inner automorphisms as $[a, x] = a^{-1}a^x$. For a general automorphism we make the following definition:

Definition 6.1. Let Q be a p-group and let χ be an automorphism of Q. For $a \in Q$ the *commutator* of a and χ is

$$[a, \chi] = a^{-1}(a\chi).$$

According to Definition 6.1 we have the following:

$$[a, \chi, \chi] = [[a, \chi], \chi] = (a^{-1}(a\chi))^{-1}(a^{-1}(a\chi))\chi = (a^{-1}\chi)a(a^{-1}\chi)(a\chi^2).$$

Note that this applies to inner automorphisms and we have

$$[a, x, x] = (a^{-1})^x a(a^{-1})^x a^{x^2}$$

for any group G with $a \in Q \leqslant G$ and $x \in N_G(Q)$.

Now we are ready to define p-stability for fusion systems.

Definition 6.2. Let $\mathcal{F}$ be a fusion system on the p-group P. Then $\mathcal{F}$ is said to be *p-stable* if for all fully $\mathcal{F}$-normalised subgroups Q of P whenever $\chi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ satisfies

$$[a, \chi, \chi] = (a^{-1}\chi)a(a^{-1}\chi)(a\chi^2) = 1$$

for all $a \in Q$, then $\chi \in O_p(\mathrm{Aut}_{\mathcal{F}}(Q))$.

Next we prove that Definition 6.2 is a generalisation of the notion of p-stability of groups to the case of fusion systems.

Theorem 6.3. *A group G is p-stable if and only if its fusion system $\mathcal{F}_P(G)$ on a Sylow p-subgroup P of G is p-stable.*

Proof. Let $Q \leqslant P$, $a \in Q$ and $x \in N_G(Q)$. Furthermore, let $\chi \in N_G(Q)/C_G(Q) = \mathrm{Aut}_{\mathcal{F}}(Q)$ be the image of x under the natural homomorphism. Then $[a, \chi, \chi] = 1$ if and only if $[a, x, x] = 1$ by the remark preceding Definition 6.2. Moreover, $xC_G(Q) \in O_p(N_G(Q)/C_G(Q)$ if and only if $\chi \in O_p(\mathrm{Aut}_{\mathcal{F}}(Q)$ as the elements and the sets coincide.

Note that as x ranges over the elements of $N_G(Q)$, its image χ ranges over the elements of $\mathrm{Aut}_{\mathcal{F}}(Q)$ and vice versa. □

Proposition 6.4. *Let $\mathcal{F}$ be a p-stable fusion system. Then all subsystems of $\mathcal{F}$ are p-stable.*

Proof. Let $\mathcal{G}$ be a subsystem of $\mathcal{F}$ on a subgroup S of P. Let Q be a subgroup of S. Assume some $\chi \in \mathrm{Aut}_{\mathcal{G}}(Q)$ satisfies $[a, \chi\chi] = 1$ for all $a \in Q$. As $\mathrm{Aut}_{\mathcal{G}}(Q) \leqslant \mathrm{Aut}_{\mathcal{F}}(Q)$, $\chi \in O_p(\mathrm{Aut}_{\mathcal{F}}(Q))$ follows. But then

$$\chi \in O_p(\mathrm{Aut}_{\mathcal{F}}(Q)) \cap \mathrm{Aut}_{\mathcal{G}}(Q) \leqslant O_p(\mathrm{Aut}_{\mathcal{G}}(Q)).$$

So $\mathcal{G}$ is p-stable. □

We can prove a theorem for fusion systems similar to Corollary 1.11:

Theorem 6.5. *Let $\mathcal{F}$ be a fusion system on a p-group P. Then $\mathcal{F}$ is p-stable if and only if $\mathcal{N}_{\mathcal{F}}(R)$ is p-stable for all non-cyclic fully $\mathcal{F}$-normalised subgroups R of P.*

Proof. One direction is clear by Proposition 6.4.

To show the converse let $Q \leqslant P$. Assume $\chi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ satisfies $[a, \chi, \chi] = 1$ for all $a \in Q$. If Q is cyclic, then $\chi \in O_p(\mathrm{Aut}_{\mathcal{F}}(Q))$ automatically follows, so we may assume Q is non-cyclic. Let $\varphi\colon Q \to R$ be an $\mathcal{F}$-isomorphism such that R is fully $\mathcal{F}$-normalised. Then $\varphi^{-1}\chi\varphi \in \mathrm{Aut}_{\mathcal{F}}(R) = \mathrm{Aut}_{\mathcal{N}_{\mathcal{F}}(R)}(R)$ satisfies $[b, \varphi^{-1}\chi\varphi, \varphi^{-1}\chi\varphi] = 1$ for all $b \in R$. As $\mathcal{N}_{\mathcal{F}}(R)$ is p-stable by assumption, $\varphi^{-1}\chi\varphi$ is contained in $O_p(\mathrm{Aut}_{\mathcal{N}_{\mathcal{F}}(R)}(R)) = O_p(\mathrm{Aut}_{\mathcal{F}}(R))$. Since $\mathrm{Aut}_{\mathcal{F}}(Q) = \varphi\,\mathrm{Aut}_{\mathcal{F}}(R)\varphi^{-1}$, it follows that $\chi \in O_p(\mathrm{Aut}_{\mathcal{F}}(Q))$. □

As mentioned before, p-soluble groups are p-stable for $p > 3$. Now we examine the relationship between p-stability and solubility for fusion systems.

Lemma 6.6. *The fusion system of $Qd(p)$ is soluble.*

Proof. The Sylow p-subgroups P of $Qd(p)$ have structure $V \rtimes C$, where V is an elementary Abelian group of rank 2 and C is a cyclic group of order p. Now, $V = O_p(Qd(p))$ and the quotient system is defined on C, a cyclic group, so the sequence

$$1 = Q_0 < Q_1 = V < Q_2 = P$$

proves the solubility of $\mathcal{F}_P(Qd(p))$. □

Proposition 6.7. *There are soluble fusion systems which are non-p-stable.*

Proof. The fusion system $\mathcal{F}_P(Qd(p))$ is soluble by Lemma 6.6 and not p-stable by Theorem 6.3. □

A counterpart of Proposition 6.7 is the following:

Theorem 6.8. *Let G be a group with Sylow p-subgroup P. If $Qd(p)$ is not involved in G, then the fusion system $\mathcal{F}_P(G)$ is soluble.*

Proof. Let G be a group not involving $Qd(p)$ and assume the theorem holds for all groups smaller than G. Let $Q = Z(J(P))$, the centre of the Thompson subgroup[2] of P. Then the normaliser $N = N_G(Q)$ controls strong fusion by Theorem B in [10, p. 1105]. It follows that $\mathcal{F}_P(G) = \mathcal{F}_P(N)$.

Therefore, $Q \trianglelefteq \mathcal{F}_P(N) = \mathcal{F}_P(G)$ and hence

$$\mathcal{F}_P(G)/Q = \mathcal{F}_{P/Q}(N/Q)$$

by Theorem 5.20 due to Stancu in [4, p. 145].

$\mathcal{F}_P(G)/Q$, being the fusion system of the $Qd(p)$-free group N/Q is soluble as $|N/Q| < |G|$. Therefore, $\mathcal{F}_P(G)$ is soluble. □

7. The maximal subgroup theorem

Our next goal is to prove a fusion theoretic version of Thompson's maximal subgroup theorem, see in [12, p. 295, Theorem 8.6.3]. For this purpose, we first state and prove a lemma that might have its own interest.

Lemma 7.1. *Let $\mathcal{N}$ be a subsystem of $\mathcal{F}$ and assume the subgroup Q of P is normal in $\mathcal{N}$. Let R be a fully $\mathcal{F}$-normalised subgroup of P that is $\mathcal{F}$-isomorphic to Q. Let $\varphi\colon N_P(Q) \to N_P(R)$ be an $\mathcal{F}$-homomorphism such that $Q\varphi = R$. Then φ induces an injective functor*

$$\Phi\colon \quad \mathcal{N} \to \mathcal{N}_{\mathcal{F}}(R)$$

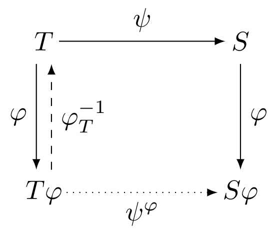
so that $\mathcal{N}$ can be embedded into $\mathcal{N}_{\mathcal{F}}(R)$.

[2] The Thompson subgroup is the subgroup of P generated by the Abelian subgroups of P of maximal order.

Proof. Note first that such a φ exists for all R (see e.g. [20, Lemma 2.2]). For an object T of $\mathcal{N}$ we define $\Phi(T) = T\varphi$. Observe that $T \leqslant N_P(Q)$ so this definition makes sense. Let now $\psi\colon T \to S$ be a morphism in $\mathcal{N}$. Then $\Phi(\psi)\colon \Phi(T) \to \Phi(S)$ is defined as

$$\Phi(\psi) = \psi^\varphi = \varphi_T^{-1}\psi\varphi_S,$$

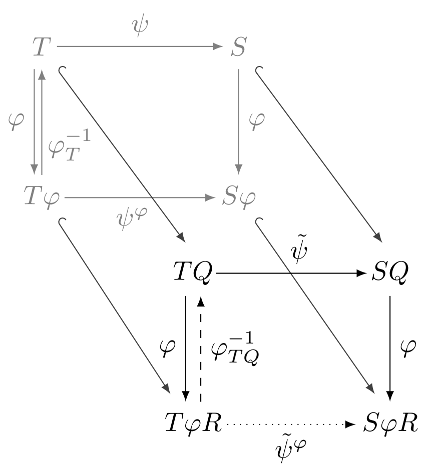
where φ_T and φ_S denote the restrictions of φ to T and S, respectively.



We claim $\Phi(\psi)$ is an $\mathcal{N}_\mathcal{F}(R)$-morphism. Indeed, as ψ is an $\mathcal{N}$-morphism and $Q \lhd \mathcal{N}$, ψ extends to a morphism $\tilde{\psi}\colon TQ \to SQ$ with $Q\tilde{\psi} = Q$. Now, $TQ \leqslant N_P(Q)$ and hence $\tilde{\psi}^\varphi$ is defined. We have $(TQ)\varphi = (T\varphi)R$ and $(SQ)\varphi = (S\varphi)R$. By construction $\tilde{\psi}^\varphi$ extends ψ^φ. Moreover,

$$R\psi^\varphi = R\varphi_Q^{-1}\psi\varphi = Q\psi\varphi = R,$$

so $\tilde{\psi}^\varphi$ extends ψ^φ in the required manner. Therefore, $\Phi(\psi)$ is indeed a morphism in $\mathcal{N}_\mathcal{F}(R)$.



It is straightforward that Φ preserves compositions and also that Φ is injective. $\square$

Theorem 7.2 *(Maximal subgroup theorem). Let $\mathcal{F}$ be a fusion system defined on the p-group P. Let $\mathfrak{Q}$ be a non-empty collection of non-trivial subgroups of P satisfying the following property:*

If $Q \in \mathfrak{Q}$, and φ: $Q \to R$ is an $\mathcal{F}$-homomorphism, then $R \in \mathfrak{Q}$.

Set

$$\mathfrak{N} = \{\mathcal{N}_{\mathcal{F}}(R) \mid 1 < R \leqslant P,\ R \text{ fully } \mathcal{F}\text{-normalised and } N_P(R) \in \mathfrak{Q}\}.$$

Assume each element of $\mathfrak{N}$ is constrained and p-stable. Then $\mathcal{N}_{\mathcal{F}}(Z(J(P)))$ is the unique maximal element of $\mathfrak{N}$.

Proof. We prove that each element of $\mathfrak{N}$ is contained in $\mathcal{M} = \mathcal{N}_{\mathcal{F}}(Z(J(P)))$. First assume $R \triangleleft P$. Then $\mathcal{N}_{\mathcal{F}}(R)$ is defined on P. As $\mathcal{N}_{\mathcal{F}}(R)$ is constrained and p-stable by assumption, it has a model L which is p-constrained, p'-reduced and p-stable. Then $C_L(O_p(L)) \subseteq O_p(L)$ and Theorem A of [10] applies. Therefore, $Z(J(P))$ is normal in L, whence $Z(J(P)) \triangleleft \mathcal{N}_{\mathcal{F}}(R)$. So $\mathcal{N}_{\mathcal{F}}(R) \subseteq \mathcal{M}$.

Let now $R \ntriangleleft P$ and assume $\mathcal{N}_{\mathcal{F}}(S) \subseteq \mathcal{M}$ for all fully $\mathcal{F}$-normalised subgroups S of P satisfying $N_P(S) \in \mathfrak{Q}$ and $|N_P(S)| > |N_P(R)|$. Now, $\mathcal{N}_{\mathcal{F}}(R)$ is defined on $N_P(R)$ and by the above argument $Z = Z(J(N_P(R))) \triangleleft \mathcal{N}_{\mathcal{F}}(R)$. Let Z^* be a fully $\mathcal{F}$-normalised subgroup of P that is $\mathcal{F}$-isomorphic to Z. Let φ: $N_P(Z) \to N_P(Z^*)$ be an $\mathcal{F}$-morphism. By Alperin's fusion theorem for fusion systems (see e.g. [4, Theorem 4.51]), there is a sequence

$$N_P(Z) = S_0 \sim S_1 \sim \ldots \sim S_t \sim S_{t+1} \subseteq N_P(Z^*)$$

of subgroups of P, there are fully $\mathcal{F}$-normalised (and essential) subgroups $L_1, \ldots, L_t$ of P such that $S_{i-1}, S_i \leqslant L_i$ for all $1 \leqslant i \leqslant t$, there are morphisms $\alpha_i \in \mathrm{Aut}_{\mathcal{F}}(L_i)$ with $S_{i-1}\alpha_i = S_i$ (for all $1 \leqslant i \leqslant t$) and there is a morphism $\sigma \in \mathrm{Aut}_{\mathcal{F}}(P)$ such that $\varphi = \alpha_1\alpha_2 \ldots \alpha_t\sigma$. Now,

$$|N_P(L_i)| \geqslant |L_i| \geqslant |S_i| = |N_P(Z)| > |N_P(R)|$$

as Z is characteristic in $N_P(R) < P$. Moreover, L_i contains S_i, a subgroup of P which is $\mathcal{F}$-isomorphic to $N_P(Z)$. Hence $L_i \in \mathfrak{Q}$. Therefore, by assumption $\mathcal{N}_{\mathcal{F}}(L_i) \subseteq \mathcal{M}$ holds for all relevant i. Observe that $\sigma \in \mathcal{M}$ is trivial. Thus

$$\varphi = \alpha_1 \ldots \alpha_t \in \mathcal{M}$$

also holds.

By Lemma 7.1 for each $\psi \in \mathcal{N}_{\mathcal{F}}(R)$ we have $\psi^\varphi \in \mathcal{N}_{\mathcal{F}}(Z^*)$, because Z is normal in $\mathcal{N}_{\mathcal{F}}(R)$. Now, $|N_P(Z^*)| \geqslant |N_P(Z)| > |N_P(R)|$ and by construction $N_P(Z^*) \in \mathfrak{Q}$. Hence $\psi^\varphi \in \mathcal{N}_{\mathcal{F}}(Z^*) \subseteq \mathcal{M}$ by assumption. Therefore,

$$\psi = \varphi_T \psi^\varphi \varphi_S^{-1} \in \mathcal{M}$$

and so $\mathcal{N}_\mathcal{F}(R) \subseteq \mathcal{M}$ which proves the theorem. □

Theorem 7.2 has the following consequence:

Proposition 7.3. *Let $\mathcal{F}$ be a fusion system and assume $\mathcal{N}_\mathcal{F}(Q)$ is constrained and p-stable for all fully $\mathcal{F}$-normalised subgroups $Q \neq 1$ of P. Then $Z(J(P)) \lhd \mathcal{F}$, so $O_p(\mathcal{F}) \neq 1$ and hence $\mathcal{F}$ is constrained and p-stable.*

Proof. Let $Z = Z(J(P))$. With the set $\mathfrak{Q} = \{1 < Q \leqslant P\}$ the conditions of Theorem 7.2 are certainly satisfied. Hence $\mathcal{N}_\mathcal{F}(Z)$ is the unique maximal element of the set

$$\mathfrak{N} = \{\mathcal{N}_\mathcal{F}(R) \mid 1 < R \leqslant P,\ R \text{ fully } \mathcal{F}\text{-normalised}\}.$$

We show $\mathcal{F} = \mathcal{N}_\mathcal{F}(Z)$. To this end, let φ: $T \to S$ be a morphism in $\mathcal{F}$. By Alperin's fusion theorem, there are subgroups

$$T = T_0 \sim T_1 \sim \ldots \sim T_t \sim T_{t+1} = T\varphi \leqslant S$$

of P and for all $i = 1, \ldots, t$, there are fully $\mathcal{F}$-normalised essential subgroups $L_i \leqslant P$ with $T_{i-1}, T_i \leqslant L_i$ and automorphisms $\tau_i \in \mathrm{Aut}_\mathcal{F}(L_i)$ with $T_{i-1}\tau_i = T_i$ and an automorphism $\sigma \in \mathrm{Aut}_\mathcal{F}(P)$ such that $\varphi = \tau_1\tau_2 \ldots \tau_t\sigma$. By assumption, for each $1 \leqslant i \leqslant t$ we have

$$\tau_i \in \mathcal{N}_\mathcal{F}(L_i) \subseteq \mathcal{N}_\mathcal{F}(Z)$$

as $L_i \neq 1$ is fully $\mathcal{F}$-normalised. On the other hand, $\sigma \in \mathcal{N}_\mathcal{F}(Z)$ trivially holds. It follows then that $\varphi \in \mathcal{N}_\mathcal{F}(Z)$ and hence $Z \lhd \mathcal{F} = \mathcal{N}_\mathcal{F}(Z)$, whence $O_p(\mathcal{F}) \supseteq Z \neq 1$. □

Concerning groups, we have the following corollary:

Corollary 7.4. *Let G be a p-stable group with Sylow p-subgroup P. Assume all p-local subgroups $N_G(Q)$ of G (with $Q \neq 1$) are p-constrained. Then the subgroup $N_G(Z(J(P)))$ controls strong fusion in P.*

Proof. Let $\mathcal{F} = \mathcal{F}_P(G)$. Then $\mathcal{N}_\mathcal{F}(Q) = \mathcal{F}_{N_P(Q)}(N_G(Q))$ is p-stable and constrained for all non-trivial fully $\mathcal{F}$-normalised subgroups of P. Hence Proposition 7.3 applies, so $O_p(\mathcal{F}) \geqslant Z(J(P)) \lhd \mathcal{F}$. As $Z(J(P))$ is fully $\mathcal{F}$-normalised, $\mathcal{F} = \mathcal{N}_\mathcal{F}(Z(J(P)))$ is the fusion system of $N_G(Z(J(P)))$, that is, $N_G(Z(J(P)))$ controls strong fusion in P. □

Remark 7.5.

(i) The assumptions in Proposition 7.3 and Corollary 7.4 are strict in the following sense: The condition that the normaliser systems (or the normalisers in the group) are p-stable cannot be omitted even if it is assumed that the normalisers are soluble (instead of being constrained). Let namely $G = L_3(3)$, P a Sylow 3-subgroup of G. Then G is a minimal simple group so that the local subgroups of G are soluble and hence so are the normaliser systems in $\mathcal{F}_P(G)$. However, the fusion system $\mathcal{F}_P(G)$ has no non-trivial normal subgroups it follows from Theorem 1.2 in [7, p. 455].

(ii) If G is p-soluble (for $p > 3$), then Theorem C in [10, p. 1105] asserts that $N_G(Z(J(P)))$ controls strong fusion in P. It follows from the results of Sections 2-3 that the fusion system of a finite simple group G is soluble if and only if $Z(J(P)) \triangleleft \mathcal{F}_P(G)$, that is, if and only if $N_G(Z(J(P)))$ controls strong fusion in P. The same is not true in general: the fusion system of $G = Qd(p)$ is soluble. A Sylow p-subgroup P of G satisfies $J(P) = P$, so $Z(J(P)) = Z(P)$ has order p. With the notation of Example 1.6, $N_G(Z(J(P))) = V$, the elementary Abelian normal subgroup of G of order p^2. As such, V does not control strong fusion in P.

8. On $Qd(p)$-free fusion systems

For groups, there is a strong connection between p-stability and not involving $Qd(p)$. A corresponding notion for fusion systems is defined in [20, Definition 1.1].

Let Q be a fully $\mathcal{F}$-normalised $\mathcal{F}$-centric subgroup of P. We examine the normaliser $\mathcal{N} = \mathcal{N}_{\mathcal{F}}(Q)$ of Q in $\mathcal{F}$. We claim $\mathcal{N}$ is constrained. Indeed, $Q \leqslant O_p(\mathcal{N})$, so

$$O_p(\mathcal{N}) \geqslant Q \geqslant C_P(Q) \geqslant C_P(O_p(\mathcal{N}))$$

as Q is $\mathcal{F}$-centric. Therefore, $\mathcal{N}$ has a model.

Definition 8.1. Let $\mathcal{F}$ be a fusion system on the p-group P. $\mathcal{F}$ is called $Qd(p)$*-free* if $Qd(p)$ is not involved in the models of $\mathcal{N}_{\mathcal{F}}(Q)$, where Q runs over the set of $\mathcal{F}$-centric fully $\mathcal{F}$-normalised subgroups of P.

We shall also call a group $Qd(p)$*-free* if it does not involve $Qd(p)$.

Remark 8.2. Though it is not stated explicitly there, it follows from [20] that a $Qd(p)$-free fusion system $\mathcal{F}$ is soluble. Indeed, Theorem B asserts that $Z(J(P))$ is normal in $\mathcal{F}$. Now, by Proposition 6.4, $\mathcal{F}/Z(J(P))$ is also $Qd(p)$-free. Since $Z(J(P))$ is non-trivial, the claim follows by induction.

As the next example shows, a soluble fusion system need not be $Qd(p)$-free.

Example 8.3. The fusion system of $Qd(p)$ is not $Qd(p)$-free: the subgroup V (as in Example 1.6) is certainly fully $\mathcal{F}$-normalised and $\mathcal{F}$-centric, its normaliser is the whole fusion system. The model of the fusion system is the group $Qd(p)$ itself, being p-constrained and p'-reduced.

Being soluble, a $Qd(p)$-free fusion system $\mathcal{F}$ is constrained and hence it has a model. By definition, a model of $\mathcal{F} = \mathcal{N}_{\mathcal{F}}(O_p(\mathcal{F}))$ is $Qd(p)$-free. Not only is a model of $\mathcal{F}$

$Qd(p)$-free, but also every group G such that $\mathcal{F} = \mathcal{F}_P(G)$ is $Qd(p)$-free, as the next result shows.

Theorem 8.4. *Let G be a group, P a Sylow p-subgroup of G and $\mathcal{F} = \mathcal{F}_P(G)$ the fusion system of G on P. Then $\mathcal{F}$ is $Qd(p)$-free if and only if G does not involve $Qd(p)$.*

In order to prove this theorem, we need some preparation.

Definition 8.5. A p-subgroup Q of G is called *p-centric* if every p-element centralising Q is contained in Q.

Note that Q is p-centric if and only if $C_P(Q) \leqslant Q$ for *all* Sylow p-subgroups P of G containing Q. In this case, $Z(Q)$ is a Sylow p-subgroup of $C_G(Q)$ and by Burnside's normal p-complement theorem it follows that $C_G(Q) = Z(Q) \times O_{p'}(C_G(Q))$.

Lemma 8.6. *Let G be a group with Sylow p-subgroup P and let $\mathcal{F} = \mathcal{F}_P(G)$ be its fusion system on P. Let furthermore Q be a fully normalised subgroup of P. Then Q is $\mathcal{F}$-centric if and only if it is p-centric.*

Proof. Q is $\mathcal{F}$-centric if and only if $C_P(Q^t) \subseteq Q^t$ holds whenever $Q^t \leqslant P$. This means that $Q \supseteq C_{P^*}(Q)$ for all Sylow p-subgroups P^* of G containing Q. This is equivalent to saying that Q is p-centric. □

Lemma 8.7. *Let G be a group, P a Sylow p-subgroup of G. Then*

$$\mathcal{F}_P(G) = \mathcal{F}_P(G/O_{p'}(G)).$$

Here, we identify $O_{p'}(G)P/O_{p'}(G)$ with P.

Proof. Denote images in $\bar{G} = G/O_{p'}(G)$ by bar. The assignment $c_{g,Q,R} \mapsto c_{\bar{g},\bar{Q},\bar{R}}$ defines a map $\mathcal{F}_P(G) \to \mathcal{F}_{\bar{P}}(\bar{G})$. We have to show it is a bijection.

We first prove it is surjective. Let $\bar{Q}$, $\bar{R} \leqslant \bar{P}$ and $\bar{g} \in \bar{G}$ such that $\bar{Q}^{\bar{g}} \leqslant \bar{R}$. Then conjugation by g maps Q into $RO_{p'}(G)$ and hence $Q^{gt} \leqslant R$ for some $t \in O_{p'}(G)$. Therefore, the image of $c_{gt,Q,R}$ is $c_{\bar{g},\bar{Q},\bar{R}}$ and surjectivity is proved.

To prove injectivity, assume $c_{\bar{g},\bar{Q},\bar{R}} = c_{\bar{h},\bar{S},\bar{T}}$. Then, first of all, $Q = S$ and $R = T$ as P maps isomorphically to $\bar{P}$. By the same reason, the operation of g and h coincides on Q. Thus $c_{g,Q,R} = c_{h,S,T}$ and injectivity is proven. □

Proposition 8.8. *Let $\mathcal{F} = \mathcal{F}_P(G)$. Furthermore, let Q be a fully $\mathcal{F}$-normalised and $\mathcal{F}$-centric subgroup of P. Then the model of $\mathcal{N}_{\mathcal{F}}(Q)$ is isomorphic to $N_G(Q)/O_{p'}(N_G(Q))$.*

Proof. We prove that the group $L = N_G(Q)/O_{p'}(N_G(Q))$ satisfies the three conditions on a model. First of all, a Sylow p-subgroup of $N_G(Q)$ is $N_P(Q)$ as Q is fully

$\mathcal{F}$-normalised. The fusion system of $N_G(Q)$ on $N_P(Q)$ is $N_{\mathcal{F}}(Q)$ by Theorem 4.27 in [4, p. 108]. Now, the fusion system of $N_G(Q)$ is the same as that of L by Lemma 8.7.

Obviously, L is p'-reduced by construction.

It only remained to show that L is p-constrained, that is,

$$C_L(O_p(L)) \leqslant O_p(L).$$

Denote the image of Q in L by $\bar{Q}$. Then $\bar{Q} \leqslant O_p(L)$ as Q is normal in $N_G(Q)$, so $C_L(O_p(L)) \leqslant C_L(\bar{Q})$.

Assume $cO_{p'}(N_G(Q))$ is contained in $C_L(\bar{Q})$ for some $c \in N_G(Q)$. Then $[c, x] \in O_{p'}(N_G(Q))$ for all $x \in Q$. But $[c, x] = x^{-c}x \in Q$, so it must be equal to 1 and hence c centralises Q. Now, $C_{N_G(Q)}(Q) = C_G(Q) = O_{p'}(C_G(Q)) \times Z(Q)$ as Q is p-centric by Lemma 8.6. As $O_{p'}(C_G(Q)) \leqslant O_{p'}(N_G(Q))$, we have $C_L(\bar{Q}) = Z(\bar{Q})$ and hence

$$C_L(O_p(L)) \leqslant C_L(\bar{Q}) \leqslant \bar{Q} \leqslant O_p(L),$$

whence the claim follows. □

Lemma 8.9. *Let G be a group. G involves $Qd(p)$ if and only if $N_G(Q)$ also does for an appropriate non-cyclic p-subgroup Q of G.*

Proof. Assume G involves $Qd(p)$, so there are $K \lhd H \leqslant G$ such that $H/K = V \rtimes S \cong Qd(p)$. Here, V is an elementary Abelian group of order p^2 and $S \cong SL_2(p)$. Let $\tilde{V}$ be a Sylow p-subgroup of the preimage of V under the natural homomorphism $H \to H/K$. Then $K\tilde{V} \lhd H$ is the preimage of V and hence $H = K\tilde{V}N_H(\tilde{V}) = KN_H(\tilde{V})$ by Frattini argument. Now,

$$Qd(p) \cong H/K = KN_H(\tilde{V})/K \cong N_H(\tilde{V})/N_H(\tilde{V}) \cap K$$

by the second isomorphism theorem. Therefore, $N_H(\tilde{V})$ and so $N_G(\tilde{V})$ involves $Qd(p)$. Finally, $\tilde{V}$ is non-cyclic as it has a non-cyclic homomorphic image V.

The other implication is clear. □

Lemma 8.10. *Let Q be a p-subgroup and P a Sylow p-subgroup of G containing a Sylow p-subgroup of $N_G(Q)$. Then any p-subgroup of G that contains $QC_P(Q)$ is p-centric.*

Proof. By construction, $C_P(Q)$ is a Sylow p-subgroup of $C_G(Q)$. Let $c \in C_G(QC_P(Q))$ be a p-element. Then c centralises Q and $C_P(Q)$, so $\langle c\rangle C_P(Q)$ is a p-group centralising Q. Hence $c \in C_P(Q) \leqslant QC_P(Q)$ by the maximality of $C_P(Q)$. □

Proposition 8.11. *Let G be a group that involves $Qd(p)$. Then $N_G(Q)$ involves $Qd(p)$ for a p-centric subgroup Q of G.*

Proof. Let $K \lhd H \leqslant G$ such that $H/K = V \rtimes S \cong Qd(p)$. By the proof of Lemma 8.9 we may assume $H \leqslant N_G(\tilde{V})$ for a p-subgroup $\tilde{V}$ of G and $W = K \cap \tilde{V}$ is a normal subgroup of H.

As $S \cong SL_2(p)$, $S = \langle x, a\rangle$ for some x, $a \in S$ such that $x^p = a^4 = 1$ and $[V, x, x] = 1$. Let moreover $\tilde{x}$ and $\tilde{a}$ be preimages of x and a under the natural homomorphism $H \to H/K$, respectively.

Let Q be a Sylow p-subgroup of $\tilde{V}C_G(\tilde{V}/W)$. Then Q is p-centric by Lemma 8.10. Let $H_1 = HC_G(\tilde{V}/W)$ and $K_1 = KC_G(\tilde{V}/W) = C_G(\tilde{V}/W)$. The latter equality holds because $[K, \tilde{V}] \subseteq K \cap \tilde{V} = W$. Observe that H_1 is a subgroup of G because H normalises both $\tilde{V}$ and W. Note that V can be identified with $\tilde{V}/W$ and we do identify them.

Now, $K_1 = \tilde{V}K_1 \lhd H_1$ and Q is a Sylow p-subgroup of K_1. Hence by Frattini argument we have

$$H_1 = N_{H_1}(Q) \cdot K_1.$$

Then $\tilde{x} = n_x \cdot k_x$ and $\tilde{a} = n_a \cdot k_a$ for appropriate elements n_x, $n_a \in N_{H_1}(Q)$ and k_x, $k_a \in K_1$.

Consider the factor group $\bar{N} = N_{H_1}(Q)/W$. By construction, $V = \tilde{V}/W \lhd \bar{N}$. Let $\bar{x}$ and $\bar{a}$ be the images under the natural homomorphism $N_{H_1}(Q) \to \bar{N}$, of n_x and n_a, respectively. Then the operations of x and $\bar{x}$ on V coincide, just as those of a and $\bar{a}$, because K_1 centralises V.

Therefore, $[V, \bar{x}, \bar{x}] = 1$, where $\bar{x} \in N_{\bar{N}}(V) = \bar{N}$. The image of $\langle \bar{x}, \bar{a}\rangle$ in $\bar{N}/C_{\bar{N}}(V)$ is isomorphic to $SL_2(p)$ and hence

$$\bar{x} \notin O_p(\bar{N}/C_{\bar{N}}(V)).$$

This means that $\bar{N}$ is not p-stable, so it involves $Qd(p)$ by Glauberman's Theorem 1.13. It follows that $N_{H_1}(Q)$ and hence $N_G(Q)$ involve $Qd(p)$. □

Now we are ready to prove the theorem.

Proof of Theorem 8.4. Assume G involves $Qd(p)$. Then $Qd(p)$ is involved in $N_G(Q)$ for some p-centric subgroup Q of P by Proposition 8.11. Observe that some conjugate of Q is fully $\mathcal{F}$-normalised and also $\mathcal{F}$-centric (the latter by Lemma 8.6). Since $Qd(p)$ has no normal p'-subgroups, it is also involved in $N_G(Q)/O_{p'}(N_G(Q))$. As this group is the model of $N_{\mathcal{F}}(Q)$ by Lemma 8.8, $\mathcal{F}$ is not $Qd(P)$-free.

For the converse, assume $\mathcal{F}$ is not $Qd(p)$-free. Then $Qd(p)$ is involved in $N_G(Q)/O_{p'}(N_G(Q))$ for some $\mathcal{F}$-centric subgroup Q of p by definition. Therefore, $Qd(p)$ is also involved in G. □

The following corollary is a slight refinement of Glauberman's Theorem 1.13:

Corollary 8.12. *The following are equivalent:*

- *All sections of G are p-stable.*
- *$N_G(Q)$ does not involve $Qd(p)$ for any p-centric p-subgroup Q of G.*

9. Section p-stability in fusion systems

We have seen in the case of groups that p-stability in itself is not enough: one needs the notion of section p-stability. Two possible definitions seem to be natural:

Definition 9.1. Let $\mathcal{F}$ be a fusion system on the p-group P. $\mathcal{F}$ is called *section p-stable* if $\mathcal{N}_{\mathcal{F}}(R)/R$ is p-stable for all fully $\mathcal{F}$-normalised subgroups R of P.

Definition 9.2. Let $\mathcal{F}$ be a fusion system on the p-group P. $\mathcal{F}$ is called *section p-stable* if the model of $\mathcal{N}_{\mathcal{F}}(R)$ is section p-stable for all $\mathcal{F}$-centric and fully $\mathcal{F}$-normalised subgroups R of P.

Clearly, Definition 9.2 is equivalent to Definition 8.1 of a $Qd(p)$-free fusion system. We show that Definitions 9.1 and 9.2 are equivalent.

Theorem 9.3. *A fusion system $\mathcal{F}$ is section p-stable according to Definition 9.1 if and only if it is section p-stable according to Definition 9.2.*

Proof. Assume $\mathcal{F}$ is section p-stable according to Definition 9.1. Let R be an $\mathcal{F}$-centric and fully $\mathcal{F}$-normalised subgroup of P. Let L be the model of $\mathcal{N}_{\mathcal{F}}(R)$ with Sylow p-subgroup $S = N_P(R)$. We have to show that $N_L(Q)/Q$ is p-stable for all subgroups of S. We can assume Q is fully $\mathcal{N}_{\mathcal{F}}(R)$-normalised. Then a Sylow p-subgroup of $N_L(Q)$ is $N_S(Q)$ and the corresponding fusion system is

$$\mathcal{F}_{N_S(Q)}(N_L(Q)) = \mathcal{N}_{\mathcal{N}_{\mathcal{F}}(R)}(Q).$$

Let $\mathcal{N} = \mathcal{N}_{\mathcal{N}_{\mathcal{F}}(R)}(Q)$. By Theorem 5.20 in [4, p. 145], we have

$$\mathcal{F}_{N_S(Q)/Q}\big(N_L(Q)/Q\big) = \mathcal{N}/Q$$

follows. In view of Theorem 6.3 we have to show that $\mathcal{N}/Q$ is p-stable.

Let Q_1 be a fully $\mathcal{F}$-normalised member of the $\mathcal{F}$-isomorphism class of Q. Then there is an $\mathcal{F}$-morphism φ: $N_P(Q) \to N_P(Q_1)$ extending an isomorphism $Q \to Q_1$ (see e.g. Lemma 2.2 in [20]). Then by Lemma 7.1, φ induces an injective functor

$$\Phi: \ \mathcal{N} \to \mathcal{N}_{\mathcal{F}}(Q_1)$$

and hence $\mathcal{N}$ can be identified with a subsystem of $\mathcal{N}_{\mathcal{F}}(Q_1)$.

We now claim that Φ induces an injective functor

$$\overline{\Phi}:\ \mathcal{N}/Q \to \mathcal{N}_{\mathcal{F}}(Q_1)/Q_1.$$

Indeed, for all objects $T \geqslant Q$ of $\mathcal{N}$ we have $\Phi(T) = T\varphi \supseteq Q\varphi = Q_1$, so we may define $\overline{\Phi}(T/Q) = T\varphi/Q_1$. Let ψ: $T \to S$ be a morphism in $\mathcal{N}$ which induces the morphism $\bar{\psi}$: $T/Q \to S/Q$ of $\mathcal{N}/Q$. Then ψ^φ induces a morphism $\overline{\psi^\varphi}$ in $\mathcal{N}_{\mathcal{F}}(Q_1)/Q_1$. What we have to show is the following: $\bar{\psi}_1 = \bar{\psi}_2$ if and only if $\overline{\psi_1^\varphi} = \overline{\psi_2^\varphi}$. In other words, $t\psi_1 Q = t\psi_2 Q$ for all $t \in T$ if and only if $(t\varphi)\psi_1^\varphi Q_1 = (t\varphi)\psi_2^\varphi Q_1$ for all $t \in T$. But this is clear by the definition of ψ_1^φ and ψ_2^φ.

Identified with a subsystem of the p-stable fusion system $\mathcal{N}_{\mathcal{F}}(Q_1)/Q_1$, the system $\mathcal{N}_{\mathcal{N}_{\mathcal{F}}(R)}(Q)/Q$ is p-stable. Hence $\mathcal{F}$ is section p-stable according to Definition 9.2.

Assume now that $\mathcal{F}$ is section p-stable according to Definition 9.2. Then $\mathcal{F}$ is $Qd(p)$-free and hence constrained by Remark 6.8. Its model G is $Qd(p)$-free, therefore section p-stable by Theorem 8.4. Now, $\mathcal{N}_{\mathcal{F}}(Q)/Q$ is the fusion system of $N_G(Q)/Q$ for all fully $\mathcal{F}$-normalised subgroups Q of P. As $N_G(Q)/Q$ is p-stable, so is $\mathcal{N}_{\mathcal{F}}(Q)/Q$. □

Proposition 9.4. *The fusion system $\mathcal{F}$ is section p-stable if and only if for all subsystems $\mathcal{G}$ of $\mathcal{F}$ and all subgroups Q of P such that $Q \triangleleft \mathcal{G}$ the quotient system $\mathcal{G}/Q$ is p-stable.*

Proof. If all subquotients are p-stable, then so are the fusion systems $\mathcal{N}_{\mathcal{F}}(R)/R$ for all fully $\mathcal{F}$-normalised subgroups R of P. Hence we only have to prove the other implication.

Let $\mathcal{F}$ be section p-stable and let $\mathcal{G}$ be an arbitrary subsystem of $\mathcal{F}$ with $Q \triangleleft \mathcal{G}$. Let Q_1 be a fully $\mathcal{F}$-normalised subgroup of P that is $\mathcal{F}$-isomorphic to Q. By the same line of arguments as in Theorem 9.3, $\mathcal{G}/Q$ is isomorphic to a subsystem of $\mathcal{N}_{\mathcal{F}}(Q_1)/Q_1$ and, as such, it is p-stable. □

10. On fusion systems on extraspecial p-groups of order p^3 and exponent p

Let E be an extraspecial group of order p^3 and exponent p. All fusion systems over E were classified by A. Ruiz and A. Viruel in [25]. A complete description of these fusion systems can be found in [4, pp. 218–226], which we shall follow here.

We examine the following questions: Which of these fusion systems are p-stable? Which of these fusion systems are section p-stable (equivalently, $Qd(p)$-free)? Which of these fusion systems are soluble?

These might be crucial in the study of p-stability since E is the Sylow p-subgroup of $Qd(p)$.

By Alperin's fusion theorem, a fusion system is completely determined by the groups $\mathrm{Aut}_{\mathcal{F}}(E)$ and $\mathrm{Aut}_{\mathcal{F}}(R)$, where R ranges over the set of essential subgroups of E. Our first observation is that essential subgroups of E in our case are precisely the radical subgroups and they are elementary Abelian of order p^2. By this, $\mathcal{F}$ is p-stable if and only if $SL_2(p)$ is not contained in $\mathrm{Aut}_{\mathcal{F}}(R)$ for any radical subgroup R of p. Having a look at the tables describing the fusion systems on E (see Tables 9.1 and 9.2 in [4, pp. 321, 323]), we obtain the result:

Proposition 10.1. *Let E be an extraspecial group of order p^3 and exponent p. Then all fusion systems defined on E are non-p-stable except for the fusion system of $G = E \rtimes H$ ($p \nmid |H|$), which is section p-stable.*

Concerning solubility, we can establish that $\mathcal{F}$ is soluble if and only if E has a non-trivial strongly closed Abelian subgroup. By Proposition 4.61 in [4, p. 129] applied to this case, Q is normal in $\mathcal{F}$ if and only if it is contained in every radical subgroup of E.

Therefore, if E has at least two radical subgroups, then the only possibility for an $\mathcal{F}$-normal subgroup is $Z(E)$. However, $SL_2(p)$ is contained in $\mathrm{Aut}_{\mathcal{F}}(R)$ for all fusion systems with at least two radical subgroups. Hence $Z(E)$ is not fixed under the action of $\mathrm{Aut}_{\mathcal{F}}(R)$, so $(Z(E)) \ntriangleleft \mathcal{F}$ in this case.

If E has exactly one radical subgroup R, then certainly $R \triangleleft \mathcal{F}$, so $\mathcal{F}$ is soluble in this case. Since the group $E \rtimes H$ with $p \nmid |H|$ is p-soluble (in which case there are no radical subgroups), its fusion system is trivially soluble.

Summarising this, we obtain:

Proposition 10.2. *Let E be an extraspecial group of order p^3 and exponent p and let $\mathcal{F}$ be a fusion system on E. Then $\mathcal{F}$ is soluble if and only if E has at most one radical subgroup, that is, if $\mathcal{F}$ is the fusion system $G \cong E \rtimes H$ with $p \nmid |H|$ or $G \cong R \rtimes (SL_2(p) \rtimes C_r)$ with $r|p-1$.*

11. Concluding remarks and questions

In Sections 2 to 4 we have shown that a finite simple group is p-stable if and only if it is section p-stable. Moreover, we have proved that a non-p-stable simple group contains a subgroup isomorphic to either $Qd(p)$ or $\widetilde{Qd}(p)$, or, if $p = 3$, $\widetilde{Qd}^{-}(3)$ or $3^2{:}(2^2{\cdot}SL_2(3))$. Also, we have determined the complete list of finite simple groups with this property by showing that one of the above groups is contained in them. We emphasise, however, that we did not try to decide when a simple group contains only one of the above groups. Also, it may contain a minimal non-p-stable group not listed here. By all these, the question naturally arises:

Question 1 *Which groups are minimal non-p-stable?*

By the results presented here, these groups have a factor group isomorphic to $Qd(p)$, but this is not a sufficient condition: Example 1.12 provides a p-stable group with $Qd(p)$ as a factor group. It might be a reachable project to determine all minimal non-p-stable groups that occur as subgroups of finite simple groups.

By an old result, if a group is soluble, then it is section p-stable, but section p-stability does not imply solubility. For fusion systems, the converse is true: if a fusion system is section p-stable, then it is soluble, but a soluble fusion system need not be section p-stable (as for the fusion system of $Qd(p)$ itself).

Also, for fusion systems of finite simple groups we have seen that p-stability and section p-stability are equivalent notions. However, this is not a general phenomenon as the fusion system of the group in Example 1.12 is p-stable but not section p-stable. Nevertheless, all of our examples of p-stable fusion systems are soluble as well. So the question arises:

Question 2 *Are there p-stable fusion systems that are not soluble?*

As soluble fusion systems have models, we can also ask:

Question 3 *Are there exotic p-stable fusion systems?*

Recall that in Section 10, the exotic ones were all non-p-stable, so we do not have any examples for that at the moment.

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