Standard Monomials and Extremal Vector Systems

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Abstract

A set system $\mathcal{F} \subseteq 2^{[n]}$ *shatters* a given set $S \subseteq [n]$ if $2^S = \{F \cap S : F \in \mathcal{F}\}$. The Sauer-Shelah lemma states that in general, $\mathcal{F}$ shatters at least $|\mathcal{F}|$ sets. A set sytstem is called *shattering-extremal* if it shatters exactly $|\mathcal{F}|$ sets. In [7] and [9] an algebraic characterization of shattering-extremal set systems was given, which offered the possibility to generalize the notion of extremality to general finite vector systems. Here we generalize the results obtained for set systems to this more general setting, and as an application, strengthen a result of Dong, Li and Zhang from [5].

Keywords: shattering-extremal set systems, standard monomials, Gröbner bases, extremal vector systems

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1 Preliminaries

Throughout this note $\mathbb{F}$ will stand for a field, and n will be a positive integer. The set $\{1, 2, \ldots, n\}$ will be referred to shortly as $[n]$ and its power set as $2^{[n]}$. We use the notation $\mathbb{F}[\mathbf{x}] = \mathbb{F}[x_1, \ldots, x_n]$ for the ring of polynomials in n variables over $\mathbb{F}$ and similarly use $f(\mathbf{x})$ for the polynomial $f(x_1, \ldots, x_n)$. If $\mathbf{w} \in \mathbb{N}^n$, we write $\mathbf{x}^{\mathbf{w}}$ for the monomial $x_1^{w_1} \ldots x_n^{w_n} \in \mathbb{F}[\mathbf{x}]$. For a subset $M \subseteq [n]$, the monomial x_M will be $\prod_{i \in M} x_i$ (and $x_\emptyset = 1$). Using this correspondence we will usually identify set systems and the corresponding collections of square-free monomials.

1.1 Shattering-extremal families

A set system *shatters* a given set $S \subseteq [n]$ if $2^S = \{F \cap S \ : \ F \in \mathcal{F}\}$. The family of subsets of $[n]$ shattered by $\mathcal{F}$ is denoted by $\mathrm{Sh}(\mathcal{F})$. The Sauer-Shelah lemma states that in general we have that $|\mathrm{Sh}(\mathcal{F})| \geq |\mathcal{F}|$ for every set system $\mathcal{F} \subseteq 2^{[n]}$. A set systems $\mathcal{F} \subseteq 2^{[n]}$ is *shattering-extremal*, or *s-extremal* for short, if it shatters exactly $|\mathcal{F}|$ sets, i.e. $|\mathcal{F}| = |\mathrm{Sh}(\mathcal{F})|$. E.g. if $\mathcal{F}$ is a *down-set*, i.e. $H \subseteq F$ and $F \in \mathcal{F}$ imply $H \in \mathcal{F}$, then $\mathcal{F}$ is s-extremal, simply because in this case $\mathrm{Sh}(\mathcal{F}) = \mathcal{F}$. The study of s-extremal set systems was initiated by Bollobás, Leader and Radcliffe in [3] and by Bollobás and Radcliffe in [4] and since then many interesting results concerning them were obtained. Anstee, Rónyai and Sali in [2] related shattering to standard monomials of vanishing ideals. Based on this, the present authors in [7] and [9] developed algebraic methods for the investigation of s-extremal families, which we recall now briefly.

1.2 Algebraic description of s-extremal families

Given some set $F \subseteq [n]$, let $v_F \in \{0, 1\}^n$ be its *characteristic vector*, i.e. the i-th coordinate of v_F is 1 if $i \in F$ and 0 otherwise. Therefore we can identify a set system $\mathcal{F} \subseteq 2^{[n]}$ with the vector system $\mathcal{V}(\mathcal{F}) = \{v_F : F \in \mathcal{F}\} \subseteq \{0, 1\}^n \subseteq \mathbb{F}^n$. One can then associate to $\mathcal{F}$ the *vanishing ideal* $I(\mathcal{F}) = I(\mathcal{V}(\mathcal{F})) = \{f \in \mathbb{F}[\mathbf{x}] : f(v_F) = 0 \text{ for every } F \in \mathcal{F}\} \trianglelefteq \mathbb{F}[\mathbf{x}]$. Note that we always have $\{x_i^2 - x_i \ : \ i \in [n]\} \subseteq I(\mathcal{F})$. The vanishing ideal of a general vector system $\mathcal{V} \subseteq \mathbb{F}^n$ can be defined similarly. For more details about vanishing ideals of finite vector systems see e.g. [9].

A total order $\prec$ on the monomials in $\mathbb{F}[\mathbf{x}]$ is a *term order*, if 1 is the minimal element of $\prec$, and $\prec$ is compatible with multiplication with monomials. One well-known and important term order is the *lexicographic (lex) order*. Here one has $\mathbf{x}^{\mathbf{w}} \prec_{\text{lex}} \mathbf{x}^{\mathbf{u}}$ iff for the smallest index k with $w_k \neq u_k$ one has $w_k < u_k$. One can build a lex order based on other orderings of the variables as well, so altogether we have $n!$ different lex orders on the monomials of $\mathbb{F}[\mathbf{x}]$. Given some term order $\prec$ and a non-zero $f \in \mathbb{F}[\mathbf{x}]$, the *leading monomial* $\text{Lm}(f)$ of f is the largest monomial (with respect to $\prec$) appearing with non-zero coefficient in the canonical form of f. For an ideal $I \trianglelefteq \mathbb{F}[\mathbf{x}]$ we denote the set of all leading monomials of polynomials in I by $\text{Lm}(I)$. A monomial is called a *standard monomial* of I if it is not a leading monomial of any $f \in I$. $\text{Sm}(I)$ denotes the set of standard monomials of I. Standard monomials have some very nice properties; among other things, in the case of vanishing ideals of finite set systems they are all square-free monomials. In general, aboutr vanishing ideals of finite vectors systems we know that they form a linear basis of the $\mathbb{F}$-vector space $\mathbb{F}[\mathbf{x}]/I$, their number equals the size of the defining vector system and for lex orders they can be computed in linear, $O(n|\mathcal{F}|k)$ time, where k is the number of different coordinates appearing (see [6]).

For an ideal $I \trianglelefteq \mathbb{F}[\mathbf{x}]$ and a term order $\prec$ a finite subset $\mathbb{G} \subseteq I$ is called a *Gröbner basis of* I with respect to $\prec$ if for every $f \in I$ there exists a $g \in \mathbb{G}$ such that $\text{Lm}(g)$ divides $\text{Lm}(f)$. $\mathbb{G}$ is a *universal Gröbner basis* if it is a Gröbner basis for every term order. Gröbner bases have many nice properties, for details the interested reader may consult e.g. [1].

The first key result in the characterization of s-extremal set systems was the algebraic description of the family of shattered sets, namely that $\text{Sh}(\mathcal{F}) = \bigcup_{\text{all term orders}} \text{Sm}(I(\mathcal{F})) = \bigcup_{\text{lex orders}} \text{Sm}(I(\mathcal{F}))$. Since the number of standard monomials of $I(\mathcal{F})$ equals $|\mathcal{F}|$ for every fixed term order, as a corollary we obtain the following proposition.

Proposition 1.1 ([7,9]) *$\mathcal{F} \subseteq 2^{[n]}$ is s-extremal iff the standard monomials of $I(\mathcal{F})$ are the same for every term/lex order.*

As mentioned earlier, for lex orders $\text{Sm}(I(\mathcal{F}))$ can be computed in linear time, however the number of possible lex orders is $n!$, and so the above result does not offer directly a method to check the extremality of a set system. However it turns out that we actually need only a significantly smaller collection of lex orders.

Theorem 1.2 ([7,9]) *Take n orderings of the variables such that for every $i \in [n]$ there is one in which x_i is the greatest element, and take the corresponding lex orders. If $\mathcal{F} \subseteq 2^{[n]}$ is not extremal, then among these we can find two term orders for which the sets of standard monomials of $I(\mathcal{F})$ differ.*

Accordingly, by computing the standard monomials for n lex orders, the extremality of a set system can be checked in $O(n^2|\mathcal{F}|)$ time.

To continue, for $i \in [n]$ define the *downshift* of $\mathcal{F} \subseteq 2^{[n]}$ by i as $D_i(\mathcal{F}) = \{F \backslash \{i\} : F \in \mathcal{F}\} \cup \{F : F \in \mathcal{F},\ i \in F,\ F \backslash \{i\} \in \mathcal{F}\}$. For indices $i_1, i_2, \ldots, i_\ell$ put $D_{i_1,i_2,\ldots,i_\ell}(\mathcal{F}) = D_{i_1}(D_{i_2}(\ldots(D_{i_\ell}(\mathcal{F}))))$. It is not hard to see that $|D_i(\mathcal{F})| = |\mathcal{F}|$ and $\mathrm{Sh}(D_i(\mathcal{F})) \subseteq \mathrm{Sh}(\mathcal{F})$, hence D_i preserves s-extremality (see e.g. [4]). Downshifts are an important tool in the study of set systems, in particular they can be used to give a possible combinatorial description of $\mathrm{Sm}(I(\mathcal{F}))$ for lexicographic term orders.

Proposition 1.3 **([7])** *Let $\mathcal{F} \subseteq 2^{[n]}$ and $\prec$ a lex order for which $x_{i_1} \succ x_{i_2} \succ \cdots \succ x_{i_n}$. Then $Sm(I(\mathcal{F})) = D_{i_n,i_{n-1},\ldots,i_1}(\mathcal{F})$.*

The results about s-extremal families also include a nice connection between s-extremal families and the theory of Gröbner bases. Given a pair of sets $H \subseteq S \subseteq [n]$ we define the polynomial $f_{S,H}(\mathbf{x}) = \mathbf{x}_H \cdot \prod_{i \in S \backslash H}(x_i - 1)$. A useful property of these polynomials is that for a set $F \subseteq [n]$ we have $f_{S,H}(v_F) \neq 0$ iff $F \cap S = H$, however much more is true.

Theorem 1.4 **([7,9])** *$\mathcal{F} \subseteq 2^{[n]}$ is s-extremal iff there are polynomials of the form $f_{S,H}$, which together with $\{x_i^2 - x_i \ : \ i \in [n]\}$ form a universal Gröbner basis of $I(\mathcal{F})$.*

We remark that in Theorem 1.4 it is enough to require a Gröbner basis of the above form for just one term order to have an s-extremal family.

1.3 Extremal vector systems

There is a usual way of generalizing the notion of shattering for a vector system $\mathcal{V} \subseteq \{0, 1, ..., k-1\}^n$, where elements of $\mathcal{V}$ are viewed as $[n] \to \{0, 1, ..., k-1\}$ functions. Now $\mathcal{V}$ *shatters* $S \subseteq [n]$ if for every function $\mathbf{g} : S \to \{0, 1, ..., k-1\}$ there exists a function $\mathbf{f} \in \mathcal{V}$ such that $\mathbf{f}|_S = \mathbf{g}$. As previously let $\mathrm{Sh}(\mathcal{V})$ denote the family of shattered sets. In the definition of extremality the Sauer-Shelah lemma played a key role, however in this case we cannot expect a similar inequality to hold. Indeed, as $\mathrm{Sh}(\mathcal{V}) \subseteq 2^{[n]}$, there are at most 2^n sets shattered, but at the same time the size of $\mathcal{V}$ can be much larger, up to k^n. This lack of a Sauer-Shelah-like inequality suggests to forget about shattering, and define extremality according to Proposition 1.1.

Proposition 1.5 **([8])** *Let $\mathcal{V} \subseteq \{0, 1, ..., k-1\}^n \subseteq \mathbb{R}^n$ be a vector system. Then $Sm(I(\mathcal{V}))$ is the same for every lex order iff $Sm(I(\mathcal{V}))$ is the same for every term order.*

Accordingly we define a finite set of vectors $\mathcal{V} \subseteq \{0, 1, ..., k-1\}^n \subseteq \mathbb{R}^n$ to be *extremal* if $\mathrm{Sm}(I(\mathcal{V}))$ is the same for every lexicographic term order, or equivalently if $\mathrm{Sm}(I(\mathcal{V}))$ is the same for every term order. Proposition 1.5 was needed to guarantee that the definition of extremality in this general setting is compatible with the special case of set systems. We remark that, although in the above definition $I(\mathcal{V})$ is considered inside $\mathbb{R}[\mathbf{x}]$, our results remain true over an arbitrary field $\mathbb{F}$ and vector systems $\mathcal{V} \subseteq \{a_1, \ldots, a_k\}^n \subseteq \mathbb{F}^n$ (see the universality property of standard monomials in [6]).

For $i \in [n]$ and for elements $\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n$ the *i-section* of $\mathcal{V}$, denoted by $\mathcal{V}_i(\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n)$ is the collection of those elements α for which $(\alpha_1, \ldots, \alpha_{i-1}, \alpha, \alpha_{i+1}, \ldots, \alpha_n) \in \mathcal{V}$. Using i-sections one can define the *downshift* at coordinate i in the general case. For any vector system $\mathcal{V} \subseteq \{0, 1, \ldots, k-1\}^n$, $D_i(\mathcal{V})$ is the unique vector system such that for every choice of elements $\alpha_1, .., \alpha_{i-1}, \alpha_{i+1}, .., \alpha_n$ the i-section of $D_i(\mathcal{V})$ is $\{0, 1, .., |\mathcal{V}_i(\alpha_1, .., \alpha_{i-1}, \alpha_{i+1}, .., \alpha_n)| - 1\}$ whenever this i-section is non-empty, and empty otherwise. For indices $i_1, i_2, \ldots, i_\ell$ let as before $D_{i_1,i_2,\ldots,i_\ell}(\mathcal{V}) = D_{i_1}(D_{i_2}(\ldots(D_{i_\ell}(\mathcal{V}))))$. Now Proposition 1.3 generalizes naturally.

Proposition 1.6 **([7])** *Let $\mathcal{V} \subseteq \{0, 1, \ldots, k-1\}^n \subseteq \mathbb{R}^n$ be a finite vector system and $\prec$ the lex order order for which $x_{i_1} \succ x_{i_2} \succ \cdots \succ x_{i_n}$. Then $Sm(I(\mathcal{V})) = D_{i_n,i_{n-1},\ldots,i_1}(\mathcal{V})$.*

Note that according to Proposition 1.6 we could have defined extremal vector systems fully combinatorially as demonstrated by the following corollary.

Corollary 1.7 *A finite vector system $\mathcal{V} \subseteq \{0, 1, \ldots, k-1\}^n$ is extremal iff $D_{\pi(n),\pi(n-1),\ldots,\pi(1)}(\mathcal{V})$ is the same for every permutation π of $[n]$.*

In [7], beside Proposition 1.6, several other results concerning this general setting were proved, however the general versions of the two main results about set systems, Theorem 1.2 and Theorem 1.4, were missing.

2 Main results [5]

A polynomial $f \in \mathbb{F}[\mathbf{x}]$ is called *degree dominated* with *dominating term* $\mathbf{x}^{\mathbf{w}}$ if it is of the form $f(\mathbf{x}) = \mathbf{x}^{\mathbf{w}} + \sum_{i=1}^{\ell} \alpha_i \mathbf{x}^{\mathbf{v}_i}$, where $\mathbf{x}^{\mathbf{v}_i} | \mathbf{x}^{\mathbf{w}}$ for every i. By basic properties of term orders we have that the dominating term of such a polynomial is also its leading term for every term order. As an example of

[5] All results of this note are part of the PhD dissertation of Tamás Mészáros. For proofs of the main results see [8].

a degree dominated polynomial one can consider any polynomial of the form $f_{S,H}$ or for $i = 1, \dots, n$ the polynomial $x_i^2 - x_i$, all of them appearing in Theorem 1.4.

Theorem 2.1 ([8]) *A finite set of vectors $\mathcal{V} \subseteq \{0, 1, ..., k-1\}^n \subseteq \mathbb{R}^n$ is extremal if and only if there is a finite family $\mathcal{G} \subseteq \mathbb{R}[\mathbf{x}]$ of degree dominated polynomials that form a universal Gröbner basis of $I(\mathcal{V})$.*

We remark that similarly as in the case of Theorem 1.4, in Theorem 2.1 it is also enough to require that $I(\mathcal{V})$ has a suitable Gröbner basis for some term order. Similarly, Theorem 1.2 also generalizes to this vector setting.

Theorem 2.2 ([8]) *Take n orderings of the variables such that for every $i \in [n]$ there is one in which x_i is the greatest element, and take the corresponding lex orders. If $\mathcal{V} \subseteq \{0, 1, ..., k-1\}^n \subseteq \mathbb{R}^n$ is not extremal, then among these we can find two term orders for which the sets of standard monomials of $I(\mathcal{V})$ differ.*

Theorem 2.2 has several interesting consequences. First of all, it means that in the definition of extremality it would have been enough to require that the family of standard monomials is the same for a particular family of n lex orders. Next, just like Theorem 1.2 for set systems, it also results an efficient, $O(n^2|\mathcal{V}|k)$ time algorithm for deciding whether a finite vector system is extremal or not. Finally, when considered over an arbitrary field $\mathbb{F}$ and for vector systems $\mathcal{V} \subseteq \{a_1, \dots, a_k\}^n \subseteq \mathbb{F}^n$, it allows a strengthening of a result by Dong, Li and Zhang from [5], where they investigated zero dimensional polynomial ideals. An ideal $I \lhd \mathbb{F}[\mathbf{x}]$ is called *zero dimensional* if the factor space $\mathbb{F}[\mathbf{x}]/I$ is a finite dimensional $\mathbb{F}$-vector space. Vanishing ideals of finite vector systems are special types of zero dimensional ideals.

A term order $\prec$ is called an *elimination order* with respect to the variable x_i if x_i is larger than any monomial from $\mathbb{F}[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$. As an example one can consider any lex order where x_i is the largest variable.

For $i \in [n]$ let $\prec_i$ be an elimination order with respect to x_i. Part (2) $\Leftrightarrow$ (3) of Theorem 4 in [5] states that if $\mathbb{F}$ has characteristic zero, then the standard monomials of any zero dimensional ideal $I \lhd \mathbb{F}[\mathbf{x}]$ are the same for every term order iff they are the same for $\prec_1, \dots, \prec_n$. We claim that (the general form of) Theorem 2.2 together with the universality property of standard monomials (see [6]) prove the same result for arbitrary fields. For this we remark, that the proof of Theorem 2.2 uses only the elimination property of lex orders and the fact that the number of standard monomials of the ideal considered is the same for every term order. Accordingly, the result remains true if we substitute the lex orders by arbitrary elimination orders with respect to the variables and the vanishing ideal $I(\mathcal{V})$ by a zero dimensional ideal I. For the second part here note that as the standard monomials form a linear basis of the $\mathbb{F}$-vector space $\mathbb{F}[\mathbf{x}]/I$, their number is the same, namely the dimension of this space, for every term order. With these observations in mind one gets the following form of Theorem 2.2, which generalizes part (2) $\Leftrightarrow$ (3) of Theorem 4 from [5] to arbitrary fields instead of fields of characteristic zero.

Theorem 2.3 **([8])** *Let $\mathbb{F}$ be an arbitrary field and for $1 \leq i \leq n$ let $\prec_i$ be an elimination order with respect to x_i. Then the standard monomials of any zero dimensional ideal $I \lhd \mathbb{F}[\mathbf{x}]$ are the same for every term order iff they are the same for $\prec_1, \ldots, \prec_n$.* $\square$

References

[1] Adams, W.W., P. Loustaunau, "An Introduction to Gröbner bases", Graduate Studies in Mathematics **3**, American Mathematical Society, 1994.

[2] Anstee, R.P., L. Rónyai, A. Sali, *Shattering News*, Graphs and Combinatorics **18** (2002), 59–73

[3] Bollobás, B., I. Leader, A.J. Radcliffe, *Reverse Kleitman Inequalities*, Proceedings of the London Mathematical Society **s3-58** (1989), 153–168

[4] Bollobás, B., A.J. Radcliffe, *Defect Sauer Results*, Journal of Combinatorial Theory, Series A **72** (1995), 189–208

[5] Dong, T., Z. Li, S. Zhang, *Finite sets of affine points with unique associated monomial order quotient bases*, Journal of Algebra and its Applications **11(2)** (2012), 1250025

[6] Felszeghy, B., B. Ráth, L. Rónyai, *The lex game and some applications*, Journal of Symbolic Computation **41** (2006), 663–681

[7] Mészáros, T., "S-extremal set systems and Gröbner bases", Diploma Thesis, Budapest University of Technology and Economics, 2010.

[8] Mészáros, T., "Algebraic Phenomena in Combinatorics: Shattering-Extremal Families and the Combinatorial Nullstellensatz", Ph.D. thesis, Central European University, Budapest, 2015.

[9] Mészáros, L. Rónyai, "Some combinatorial application of Gröbner bases", In: F. Winkler, Algebraic Informatics, CAI 2011, Lecture Notes in Computer Science **6742**, Springer 2011., 64–83