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Realisability of p-stable fusion systems

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ABSTRACT

The aim of this paper is to investigate p-stable fusion systems, where p is an odd prime. We examine realisable fusion systems and prove a generalisation of a result of G. Glauberman. Then we prove that p-stability is determined by the normaliser systems of centric radical subgroups. Finally, we prove that all p-stable fusion systems are realisable provided there exists a stable p-functor.

0. Introduction

Throughout this paper, p denotes an odd prime. All groups considered in this paper are finite. The concept of p-stability was originally defined for groups by D. Gorenstein and J.H. Walter (see [8]). The definition used now is due to G. Glauberman (see [7]). In a joint work with Professor A.E. Zalesski (see [10]), we generalised this concept to

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fusion systems. All fusion systems are assumed to be saturated. The aim of this paper is to investigate further properties of p-stable fusion systems.

In Section 1, we give the main definitions and preliminary results that we need later. In Section 2, we investigate realisable p-stable fusion systems and prove a generalisation of Theorem B of G. Glauberman (see [6]):

Theorem 1. *Let $p > 3$ and let G be a p-stable group. Then $N_G(Z(J(P)))$ controls strong fusion in P.*

Here, $J(P)$ denotes the Thompson subgroup of P, the subgroup generated by the Abelian subgroups of maximal order.

Section 3 is devoted to the local properties of p-stability. We prove there the following:

Theorem 2. *Let $\mathcal{F}$ be a fusion system defined on P. Then $\mathcal{F}$ is p-stable if and only if $\mathcal{N}_{\mathcal{F}}(Q)$ is p-stable for all fully normalised, centric, radical subgroups Q of P.*

In Section 4 we introduce the concept of a stable p-functor and show the following:

Theorem 3. *If there exists a stable p-functor, then every p-stable fusion system is realisable.*

1. Preliminaries

In this section we introduce the main definitions and results used later in this article. We begin with p-stability and continue with fusion systems. The following definition is due to G. Glauberman, see [7].

Definition 1.1. A finite group G is called p-stable if for all p-subgroups Q of G and all elements $x \in N_G(Q)$ whenever

$$[Q, x, x] = 1,$$

then the coset

$$xC_G(Q) \in O_p(N_G(Q)/C_G(Q)).$$

The smallest group which is not p-stable is the group $Qd(p) = V \rtimes SL_2(p)$, where $V \cong C_p^2$ and $SL_2(p)$ acts in the natural way. It is shown in [6] that all sections of a group are p-stable if and only if the group does not involve $Qd(p)$. A group is called $Qd(p)$-free if it does not involve $Qd(p)$. By Glauberman's result a $Qd(p)$-free group is p-stable. The converse is false; Professor O. Yakimova has called our attention to the following group: There is a uniserial $\mathbb{F}_pSL_2(p)$-module U of dimension $p+1$ with a factor isomorphic to the natural $SL_2(p)$-module. Then the semidirect product of $SL_2(p)$ with U is a p-stable

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group possessing a factor group isomorphic to $Qd(p)$. For more details, see Example 1.12 in [10].

Definition 1.2. Let G be a finite group with Sylow p-subgroup P. Let $H \leqslant G$. Then H is said to *control strong fusion* in P if for all subgroups Q of P and all elements $g \in G$ such that $Q^g \leqslant P$ there exists an element $h \in H$ and $c \in C_G(Q)$ with $g = ch$.

Note that by Definition 1.2, the group homomorphism $c_g\colon Q \to Q^g$ defined by $x \mapsto x^g$ coincides with the homomorphism c_h defined in a similar way.

For a subgroup $Q \leqslant P$, it is often said in the literature that 'Q controls strong fusion in P' for '$N_G(Q)$ controls strong fusion in P.' To avoid confusion, we always use control of fusion in the sense as in Definition 1.2.

The notion of saturated fusion system has now become standard. For the main definitions, we refer to [4], [12] or [1]. In this paper, all *fusion systems are saturated*, so we omit the adjective 'saturated'.

For a morphism $\chi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ and an element $a \in Q$ we let $[a, \chi] = a^{-1}(a\chi)$. (Note that morphisms are written from the *right* as in [4].)

Definition 1.3. Let $\mathcal{F}$ be a fusion system on the p-group P. Then $\mathcal{F}$ is called *p-stable* if for all $Q \leqslant P$ and for all $\chi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ whenever

$$[Q, \chi, \chi] = 1,$$

then $\chi \in O_p(\mathrm{Aut}_{\mathcal{F}}(Q))$.

A stronger notion for both groups and fusion systems is *section p-stability* as defined in [10]. In both cases it turns out to be equivalent to $Qd(p)$-freeness (the latter having been defined for fusion systems in [11]).

The fusion system of a group G on a Sylow p-subgroup P is denoted by $\mathcal{F}_P(G)$. Consider the group $\bar{G} = G/O_{p'}(G)$. Then $\bar{P} = PO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $\bar{G}$. Observe that P and $\bar{P}$ are isomorphic, so we may and do identify them. Then the fusion systems $\mathcal{F}_P(G)$ and $\mathcal{F}_{\bar{P}}(\bar{G})$ coincide (for the details see e.g. Lemma 8.7 in [10, p. 290]). A group is called *p'-reduced* if $O_{p'}(G) = 1$.

A fusion system $\mathcal{F}$ is said to be *realisable* if it is the fusion system of some group G. By the above paragraph G may be assumed to be p'-reduced.

The largest subgroup of P that is normal in the fusion system $\mathcal{F}$ is denoted by $O_p(\mathcal{F})$. We call $\mathcal{F}$ *constrained* if $C_P(O_p(\mathcal{F})) \leqslant O_p(\mathcal{F})$. Each constrained fusion system is realisable. More precisely, there is a p-constrained, p'-reduced group G with Sylow p-subgroup P such that $\mathcal{F} = \mathcal{F}_P(G)$, see [3]. Such a group is called *model* of $\mathcal{F}$.

By a result of Aschbacher (see [2]), each soluble fusion system is constrained and hence realisable. In [11] it is shown that every $Qd(p)$-free fusion system is soluble (and hence constrained and realisable).

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2. On realisable p-stable fusion systems

In this section we prove some theorems concerning realisable fusion systems.

Proposition 2.1. *Let G be a finite group and let P be a Sylow p-subgroup of G. Set $\mathcal{F} = \mathcal{F}_P(G)$. Assume $\mathcal{F}$ is p-stable. Then $O_p(\mathcal{F}) \neq 1$.*

Proof. Assume to the contrary and let G be a minimal counterexample to the statement. Observe that the fusion systems of G and of $G/O_{p'}(G)$ coincide. Hence $O_{p'}(G) = 1$ otherwise the factor group would be a smaller counterexample. If G is simple, then it is $Qd(p)$-free by Theorem 2 in [10, p. 254]. Therefore, it is soluble and hence G is not a counterexample. So G is non-simple.

Let $1 < N \lhd G$ be a proper normal subgroup of G and let $Q = N \cap P$ be a Sylow p-subgroup of N. Then $G = N \cdot N_G(Q)$ by the Frattini argument. Therefore, $Q \neq N$ since otherwise $Q \lhd \mathcal{F}$ which is impossible. Similarly, $Q \neq 1$ as it would imply $O_{p'}(G) \neq 1$.

Now, $\mathcal{F}_Q(N)$ is p-stable being a subsystem of $\mathcal{F}$. Since $N < G$, it follows that $O_p(\mathcal{F}_Q(N)) \neq 1$. On the other hand, $\mathcal{F}_Q(N)$ is weakly normal in $\mathcal{F}$ (see Lemma 5.32 in [4, p. 151]) and hence $O_p(\mathcal{F}_Q(N)) \leqslant O_p(\mathcal{F})$ by Proposition 5.47 in [4, p. 158]. Then $O_p(\mathcal{F}) \neq 1$, a contradiction. □

Lemma 2.2. *Let Z be a central subgroup of G. Assume G/Z is p-stable. Then so is G.*

Proof. Although in [5, p. 83] an earlier definition of p-stability is used, a slight modification of the proof gives the following result: an arbitrary group H is p-stable if and only if $H/O_{p'}(H)$ is p-stable. Then a similar statement follows with an arbitrary p' normal subgroup (instead of $O_{p'}(H)$).

Put $Z = Z_p \times Z_{p'}$. By the above, G/Z_p is p-stable if and only if so is $G/Z \cong (G/Z_p)/(Z/Z_p)$. Therefore, we may assume Z is a p-group.

Let $\bar{G} = G/Z$ ad let us denote images under the natural homomorphism $G \to \bar{G}$ by bars. Let Q be a p-subgroup of G and let $x \in N_G(Q)$ be a p-element such that $[Q, x, x] = 1$. Then $[\bar{Q}, \bar{x}, \bar{x}] = \bar{1}$ and hence

$$\bar{x} C_{\bar{G}}(\bar{Q}) \in O_p(N_{\bar{G}}(\bar{Q})/C_{\bar{G}}(\bar{Q}))$$

as $\bar{G}$ is p-stable. Now, $N_{\bar{G}}(\bar{Q}) = \overline{N_G(ZQ)}$ by the homomorphism theorem. Observe that $N_G(ZQ) \geqslant N_G(Q)$ and $C_G(ZQ) = C_G(Q)$ since Z is central. Moreover, $\overline{C_G(Q)} \leqslant C_{\bar{G}}(\bar{Q})$. Let C be the full preimage of $C_{\bar{G}}(\bar{Q})$ under the natural homomorphism, that is,

$$C = \{g \in G \mid [Q, g] \subseteq Z\}.$$

Let $A = N_G(Q)/C_G(Q)$ and $B = N_{\bar{G}}(\bar{Q})/C_{\bar{G}}(\bar{Q})$. Then there is a natural homomorphism

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$$\Psi\colon A \to B,$$

whose kernel is $C/C_G(Q)$.

By construction,

$$\bar{x}C_{\bar{G}}(\bar{Q}) \in O_p(B) \cap \Psi(A) \leqslant O_p(\Psi(A)).$$

We claim $C/C_G(Q)$ is a p-group. To see this, let $g \in C$. Then for each $q \in Q$, there is some $z \in Z$ such that

$$q^g = zq.$$

Then

$$q^{g^2} = (zq)^g = zq^g = z^2q$$

and by induction $q^{g^j} = z^jq$. Then $g^{|Z|} \in C_G(Q)$ as $z \in Z$. Since, by assumption, Z is a p-group, the claim follows.

Therefore, $\Psi(O_p(A)) = O_p(\Psi(A))$ and hence $xC_G(Q) \in O_p(A)$ whence the lemma. □

Proposition 2.3. *Let $p > 3$. Let $\mathcal{F} = \mathcal{F}_P(G)$. Assume $\mathcal{F}$ is p-stable. Then $\mathcal{F}$ is constrained.*

Proof. Let G be a minimal counterexample to the statement. Set $Q = O_p(\mathcal{F})$. Then $Q \neq 1$ by Proposition 2.1. Consider the centraliser subsystem $\mathcal{C} = \mathcal{C}_{\mathcal{F}}(Q)$. Then $\mathcal{C} = \mathcal{F}_{C_P(Q)}(C_G(Q))$ by Theorem 4.27 in [4, p. 108].

Assume $\mathcal{C} \subsetneq \mathcal{F}$. Then, by assumption, $\mathcal{C}$ is constrained. Now, $\mathcal{C}$ is weakly normal in $\mathcal{F}$ and hence

$$O_p(\mathcal{C}) = C_P(Q) \cap Q = Z(Q)$$

by [4, Proposition 5.47]. Then $C_{C_P(Q)}(Z(Q)) \subseteq Z(Q)$ follows by the constraint of $\mathcal{C}$. Since

$$C_{C_P(Q)}(Z(Q)) = C_P(Q),$$

we have $C_P(Q) \subseteq Q$ contradicting the assumption that $\mathcal{F}$ is not constrained.

Therefore, $\mathcal{F} = \mathcal{C}$ and hence $P = C_P(Q)$, so $Q \leqslant Z(P)$. Furthermore, $G = C_G(Q)$ as otherwise $C_G(Q)$ would be a smaller counterexample. Now, G is not simple, otherwise it would be $Qd(p)$-free and hence constrained, see [10] and [11]. Let N be a maximal normal subgroup of G. Then $Q \leqslant N$ since otherwise NQ would be a larger normal subgroup of G.

Let $R = N \cap P$ be a Sylow p-subgroup of N and let $\mathcal{N} = \mathcal{F}_R(N)$. Then $\mathcal{N}$ is weakly normal in $\mathcal{F}$ and hence $O_p(\mathcal{N}) = Q \cap R = Q$.

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By the minimality of G, $\mathcal{N}$ is constrained, so $C_R(Q) \subseteq Q$. On the other hand, $C_R(Q) = R$ as Q is central in G. Thus $R = Q$ follows, so Q is a central Sylow p-subgroup of N. Hence by Burnside's normal p-complement theorem $N = K \times Q$ follows, where K is a p'-group. Then $K \lhd G$, whence $K = 1$ and $N = Q$ is a p-group.

Therefore, $\bar{G} = G/Q$ is simple and Q is a central p-subgroup of G. Furthermore, $\bar{G}$ is non-Abelian as G cannot be soluble. In particular, $G = QG'$. We claim G' is a counterexample to the statement. Let $P_1 = P \cap G'$ and $Q_1 = Q \cap G'$ so that $P = QP_1$ by construction. Moreover, $\mathcal{F}' = \mathcal{F}_{P_1}(G')$ is p-stable. Observe that $Q_1 = O_p(\mathcal{F}')$ since $\mathcal{F}'$ is weakly normal in $\mathcal{F}$. Q_1 is central in G' and hence $\mathcal{F}'$ is not constrained unless $Q_1 = P_1$. This is, however, impossible since then G' would have a normal p-complement, which would be a normal p'-subgroup in G.

Therefore, $G = G'$ and hence G is a stem extension of the non-Abelian simple group $\bar{G}$ by the p-group Q. Looking at the list of finite simple groups and their Schur multipliers, we obtain $\bar{G} \cong PSL_n(q)$ or $PSU_n(q)$, where $p | \gcd(n, q-1)$ or $p | \gcd(n, q+1)$, respectively. Then G is a central factor of $\tilde{G} = SL_n(q)$ or $SU_n(q)$. By Lemma 2.2, $\tilde{G}$ is p-stable as G is so. However, by [10], $\bar{G}$ is p-stable if an only if so is $\tilde{G}$. Hence $\bar{G}$ is p-stable. Theorem 1 in [10] then implies that the fusion system $\bar{\mathcal{F}}$ of $\bar{G}$ is soluble. Now, $\bar{\mathcal{F}} = \mathcal{F}/Q$, so $\mathcal{F}$ is soluble and hence constrained, a contradiction. □

Proposition 2.3 enables us to prove a generalisation of Theorem B of Glauberman in [6, p. 1105]:

Theorem 2.4 *(Glauberman 1968). Let G be a $Qd(p)$-free group. Then $N_G(Z(J(P)))$ controls strong fusion in P.*

We now prove that for $p > 3$ the condition on G to be $Qd(p)$-free can be replaced by the weaker condition of p-stability.

Theorem 2.5. *Let $p > 3$ and let G be a p-stable group. Then $N_G(Z(J(P)))$ controls strong fusion in P.*

Proof. Let $\mathcal{F} = \mathcal{F}_P(G)$. Then $\mathcal{F}$ is p-stable and, by Proposition 2.3, constrained. Therefore, it has a model L by Proposition C in [3]. By definition, L is p'-reduced and p-constrained and $\mathcal{F} = \mathcal{F}_L(P)$. Moreover, L is p-stable by Theorem 6.3 in [10] since $\mathcal{F}$ is p-stable. Then Theorem A in [6, p. 1105] applies and $Z(J(P)) \lhd L$. Hence $Z(J(P))$ is normal in $\mathcal{F} = \mathcal{F}_P(L) = \mathcal{F}_P(G)$. Therefore, $\mathcal{F}$ is the fusion system of $N_G(Z(J(P)))$ on P (see Theorem 4.27 in [4, p. 108]). This means that for any subgroup Q of P and any element $g \in G$ such that $Q^g \subseteq P$, there is some $n \in N_G(Z(J(P)))$ such that the conjugation action $c_g\colon Q \to Q^g$ coincides with $c_n\colon Q \to Q^n = Q^g$. Hence $c = gn^{-1} \in C_G(Q)$, that is, $g = cn$. Therefore, $N_G(Z(J(P)))$ controls strong fusion in P. □

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3. Local subgroups and p-stability

In [10] it has been shown that a fusion system is p-stable if and only if the local subsystems $\mathcal{N}_{\mathcal{F}}(Q)$ are p-stable. Now we prove a refinement of this theorem.

Theorem 3.1. *Let $\mathcal{F}$ be a fusion system defined on P. Then $\mathcal{F}$ is p-stable if and only if $\mathcal{N}_{\mathcal{F}}(Q)$ is p-stable for all fully normalised, centric, radical subgroups Q of P.*

Proof. If $\mathcal{F}$ is p-stable, then so are all subsystems of $\mathcal{F}$ (see [10, Proposition 6.4]) so we only have to show the 'if' part.

Assume $\mathcal{F}$ is not p-stable. Then by definition of p-stability there is a fully $\mathcal{F}$-normalised subgroup S of P and an $\mathcal{F}$-automorphism χ of S such that $[S, \chi, \chi] = 1$ and $\chi \notin O_p(\mathrm{Aut}_{\mathcal{F}}(S))$.

Let $Q = SC_P(S)$ and let Q' be a fully normalised $\mathcal{F}$-conjugate of Q. Let φ: $Q \to Q'$ be an $\mathcal{F}$-isomorphism that extends to an $\mathcal{F}$-morphism $\tilde{\varphi}$: $N_P(Q) \to N_P(Q')$. Such a morphism exists by [12, Lemma 2.6]. Let $S' = S\varphi$. Observe that Q is normal in $N_P(S)$ and hence $\tilde{\varphi}$ maps $N_P(S)$ into $N_P(Q')$. Note that this image is contained in $N_P(S')$. Therefore, S' is fully normalised and $N_P(S') \subseteq N_P(Q')$.

Now, $Q' = S'C_P(S')$ is centric (see Lemma 4.42 in [4, p. 117]. Let $\mathcal{N} = \mathcal{N}_{\mathcal{F}}(Q')$. We claim $\mathcal{N}$ is not p-stable. Let $\chi' = \varphi^{-1}\chi\varphi \in \mathrm{Aut}_{\mathcal{F}}(S')$. Then $[S', \chi', \chi'] = 1$ and $\chi' \notin O_p(\mathrm{Aut}_{\mathcal{F}}(S'))$. This shows that $\mathcal{N}$ is not p-stability once we prove

$$\mathrm{Aut}_{\mathcal{N}}(S') = \mathrm{Aut}_{\mathcal{F}}(S').$$

Since S' is fully normalised and hence receptive, each $\mathcal{F}$-automorphism ψ of S' extends to $Q' = S'C_P(S')$ as the latter is certainly contained in N_ψ. Hence by definition ψ is a morphism in $\mathcal{N}$ and the claim follows.

To finish the proof, we have to show that there exists a fully normalised, centric, radical subgroup R of P such that $\mathcal{N}_{\mathcal{F}}(R)$ is not p-stable. Let L be a model of $\mathcal{N}$, which exists since Q' is centric. In the proof of Proposition 6.1 of [11] it is shown that L is contained in a model M of the normaliser system of some fully normalised, centric, radical subgroup R of P. Since $\mathcal{N}$ is not p-stable, L and hence its overgroup M are not p-stable as well (see Proposition 1.8. and Theorem 6.3 in [10]). Then $\mathcal{N}_{\mathcal{F}}(R)$ is not p-stable and the theorem is proven. □

Recall that Alperin's fusion theorem has several formulations. In one of them the existence of a series of p-centric radical subgroups of P is stated while in another one that of a series of essential subgroups. The question naturally arises whether it is enough to test p-stability and $Qd(p)$-freeness on the normaliser systems of essential subgroups and P rather than of centric, radical subgroups. We formulate this problem below.

Problem 3.2. Let $\mathcal{F}$ be a fusion system defined on P. Is $\mathcal{F}$ p-stable if (and only if) $\mathcal{N}_{\mathcal{F}}(P)$ and $\mathcal{N}_{\mathcal{F}}(E)$ are p-stable for all essential subgroups E of P?

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4. Realisability and p-stability

We now investigate the relationship of p-stability and realisability of fusion systems. To state our result concerning this relationship we need some preparation.

Definition 4.1. A *positive characteristic p-functor* is a mapping W defined on the class of finite p-groups that assigns to each p-group P a non-trivial characteristic subgroup $W(P)$ of P with the property that for each isomorphism φ: $P \to P'$ the image of $W(P)$ under φ is $W(P')$.

A positive characteristic p-functor W is called *Glauberman functor* if it has the additional property that $W(P)$ is normal in each p'-reduced, p-constrained group G which does not involve $Qd(p)$ and whose Sylow p-subgroup is P.

In [6, Theorem A] Glauberman shows that the assignment ZJ: $P \mapsto Z(J(P))$ is a Glauberman functor. This functor has another interesting property: if G is a p-stable and p-constrained group with Sylow p-subgroup P, then $N_G(Z(J(P)))$ controls strong fusion in P (see [6, Theorem C]).

Other examples of Glauberman functors are K_∞ and K^∞. For the definition, see [7]. These functors satisfy $C_P(K_\infty(P)) \subseteq K_\infty(P)$ and $C_P(K^\infty(P)) \subseteq K^\infty(P)$ (see e.g. [9, Lemma 8.5]). It is not known (at least to us), however, whether $N_G(K_\infty)$ or $N_G(K^\infty)$ controls strong fusion in every p-stable and p-constrained group G with Sylow p-subgroup P.

It is also not known whether there exists a positive characteristic p-functor that enjoys both of the properties mentioned in the previous two paragraphs.

Definition 4.2. We call a positive characteristic p-functor W *stable p-functor* if

- $C_P(W(P)) \subseteq W(P)$ for all P; and
- $N_G(W(P))$ controls strong fusion in P whenever G is a p-stable and p-constrained group with Sylow p-subgroup P.

Problem 4.3. Does there exist a stable p-functor?

We now prove the main result of this section.

Theorem 4.4. *Assume there exists a stable p-functor W. Then every p-stable fusion system is realisable.*

Proof. Assume to the contrary and let $\mathcal{F}$ be a minimal counterexample to the statement. If $W(P) \lhd \mathcal{F}$, then $\mathcal{F}$ is constrained (as the centric subgroup $W(P) \leqslant O_p(\mathcal{F})$). Therefore, $\mathcal{F}$ is realisable. So we can assume $W(P) \ntriangleleft \mathcal{F}$ and hence

$$\mathcal{N} = \mathcal{N}_{\mathcal{F}}(W(P)) \subsetneq \mathcal{F}.$$

Then by Alperin's fusion theorem (see Theorem 4.51 in [4, p. 121]), there exists a fully normalised essential subgroup R of P such that

$$\mathrm{Aut}_{\mathcal{N}}(R) \lneq \mathrm{Aut}_{\mathcal{F}}(R).$$

Let $P_1 = N_P(R)$. Assume $P_1 = P$. Since R is centric, $\mathcal{N}_{\mathcal{F}}(R)$ is realisable and hence $W(P) \trianglelefteq \mathcal{N}_{\mathcal{F}}(R)$. Thus $\mathcal{N}_{\mathcal{F}}(R) \subseteq \mathcal{N}$. But then

$$\mathrm{Aut}_{\mathcal{F}}(R) = \mathrm{Aut}_{\mathcal{N}_{\mathcal{F}}(R)}(R) \leqslant \mathrm{Aut}_{\mathcal{N}}(R),$$

a contradiction.

Therefore, $P_1 < P$. Now, $\mathcal{N}_1 = \mathcal{N}_{\mathcal{F}}(R)$ is realisable and p-stable because it is a proper subsystem of $\mathcal{F}$ and $\mathcal{F}$ is a minimal counterexample. Let L be a model of $\mathcal{N}_{\mathcal{F}}(R)$. Then $N_L(W(P_1))$ controls strong fusion in P_1 and thus $W(P_1) \trianglelefteq \mathcal{N}_{\mathcal{F}}(R)$. Let $W_1 = W(P_1)$ and $P_2 = N_P(W_1)$. Then $P_2 \geqslant N_P(P_1) > P_1$ as $P_1 < P$ and W_1 is characteristic in P_1. Let $\mathcal{N}_2 = \mathcal{N}_{\mathcal{F}}(W_1)$. Then

$$\mathcal{F} \supseteq \mathcal{N}_2 \supsetneq \mathcal{N}_{\mathcal{F}}(R)$$

because W_1 is normal in $N_{\mathcal{F}}(R)$ and $\mathcal{N}_2$ is defined on a larger subgroup than $N_{\mathcal{F}}(R)$. If $\mathcal{F} \neq \mathcal{N}_2$, then $\mathcal{N}_2$ is realisable and hence it is constrained by Proposition 2.3. Therefore, $W_2 = W(P_2) \trianglelefteq \mathcal{N}_2$ as W is a stable p-functor.

Proceeding similarly, for each integer $i > 1$ we define $P_i = N_P(W_{i-1})$, $W_i = W(P_i)$, and $\mathcal{N}_i = \mathcal{N}_{\mathcal{F}}(W_{i-1})$. Then $P_i \geqslant N_P(P_{i-1})$ for each i and hence there is some t such that $P_{t-1} < P_t = P$. Furthermore, if $\mathcal{N}_i \subsetneq \mathcal{F}$, then $\mathcal{N}_i$ is realisable and W_i is normal in $\mathcal{N}_i$ by repeating the above argument for a general i instead of $i = 2$. Therefore, $\mathcal{N}_{i+1} \supseteq \mathcal{N}_i$. Note that this containment is proper if $P_i < P$. Summarising the above, we have:

$$\mathcal{N}_1 \subsetneq \mathcal{N}_2 \subsetneq \ldots \subsetneq \mathcal{N}_t$$

and

$$P_1 < P_2 < \ldots < P_t = P.$$

Now, $\mathcal{N}_t$ is defined on $P_t = P$. If $\mathcal{N}_t \neq \mathcal{F}$, then $W(P) = W_t$ is normal in $\mathcal{N}_t$ by the above argument. So $\mathcal{N}_t \subseteq \mathcal{N}_{\mathcal{F}}(W(P)) = \mathcal{N}$ in this case. But then

$$\mathrm{Aut}_{\mathcal{F}}(R) = \mathrm{Aut}_{\mathcal{N}_1}(R) \leqslant \mathrm{Aut}_{\mathcal{N}_t}(R) \leqslant \mathrm{Aut}_{\mathcal{N}}(R),$$

which contradicts the choice of R. Hence we can conclude $\mathcal{N}_t = \mathcal{F}$, so $W_{t-1} \trianglelefteq \mathcal{F}$.

Therefore, $\mathcal{C} = \mathcal{C}_{\mathcal{F}}(W_{t-1})$ is a weakly normal subsystem of $\mathcal{F}$. Let $O = O_p(\mathcal{C})$. Then $O = O_p(\mathcal{F}) \cap C_P(W_{t-1})$. Being the intersection of two strongly $\mathcal{F}$-closed subgroups, O itself is strongly $\mathcal{F}$-closed. Then by Theorem 9.1 in [12], $O \trianglelefteq \mathcal{F}$.

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Since W is a stable p-functor, $C_{P_{t-1}}(W_{t-1}) \leqslant W_{t-1} < P_{t-1}$. Hence $\mathcal{C}$ is defined on a proper subgroup of P and, as such, it is a proper subsystem of $\mathcal{F}$. By assumption $\mathcal{C}$ is then realisable and p-stable, whence constrained by Proposition 2.3.

Let $Q = W_{t-1} \cdot O$. Being a product of normal subgroups of $\mathcal{F}$, $Q \triangleleft \mathcal{F}$. Now,

$$C_P(Q) = C_P(W_{t-1}) \cap C_P(O) = C_{C_P(W_{t-1})}(O) \leqslant O \leqslant Q.$$

This means that Q is a centric normal subgroup of $\mathcal{F}$, whence $\mathcal{F}$ is constrained and hence realisable, contradicting the assumption. □

References

[1] M. Aschbacher, R. Kessar, B. Oliver, Fusion Systems in Algebra and Topology, LMS Lecture Notes, vol. 391, Cambridge University Press, 2011.

[2] M. Aschbacher, The Generalized Fitting Subsystem of a Fusion System, Mem. Amer. Math. Soc., vol. 209, AMS, 2011.

[3] C. Broto, N. Castellana, J. Grodal, R. Levi, B. Oliver, Subgroup families controlling p-local finite groups, Proc. Lond. Math. Soc. 91 (2005) 325–354.

[4] D.A. Craven, The Theory of Fusion Systems: An Algebraic Approach, Cambridge Stud. Adv. Math., vol. 131, Cambridge University Press, 2011.

[5] T.M. Gagen, Topics in Finite Groups, Lond. Math. Soc. Lecture Notes Ser., vol. 16, Cambridge University Press, 1976.

[6] G. Glauberman, A characteristic subgroup of a p-stable group, Canad. J. Math. 20 (1968) 1101–1135.

[7] G. Glauberman, Global and local properties of finite groups, in: M.B. Powell, G. Higman (Eds.), Finite Simple Groups, Proc. of an Instructional Conference Organized by the LMS, Academic Press, 1971, pp. 1–63.

[8] D. Gorenstein, J.H. Walter, On the maximal subgroups of finite simple groups, J. Algebra 1 (1964) 168–213.

[9] B. Huppert, N. Blackburn, Finite Groups III, Grundlehren Math. Wiss., vol. 243, Springer Verlag, Berlin, Heidelberg, New York, 1982.

[10] L. Héthelyi, M. Szőke, A.E. Zalesski, On p-stability in groups and fusion systems, J. Algebra 492 (2017) 253–297.

[11] R. Kessar, M. Linckelmann, ZJ-theorems for fusion systems, Trans. Amer. Math. Soc. 360 (2) (2008) 3093–3106.

[12] Markus Linckelmann, Introduction to fusion systems, in: Group Representation Theory, EPFL, Press, 2007, pp. 79–113.

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