Empirical portfolio selections

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Portfolio selections

- static portfolio selection (single period)

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- constantly rebalanced portfolio

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- multi-asset, multi-period
- empirical (nonparametric statistics, machine learning)

Growth rate

investment in the stock market

d assets

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asymptotic growth rate

$$W^{(j)} = \lim_{n\to\infty} W_n^{(j)} = \lim_{n\to\infty} \frac{1}{n} \ln S_n^{(j)}$$

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we can do much better, applying dynamic portfolio selection

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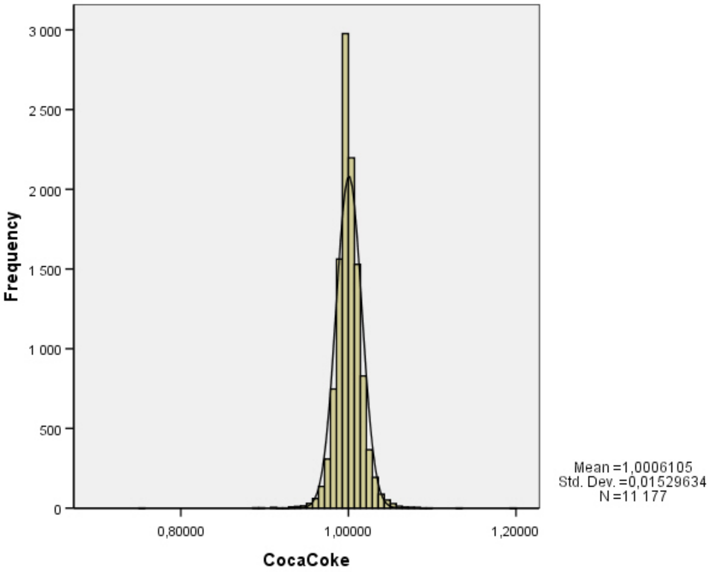
For NYSE data

$$0.7 \leq x_i^{(j)} \leq 1.2$$


3 000
2 500
2 000
1 500
1 000
500
0
Frequency
0,80000
1,00000
1,20000
CocaCoke
Mean =1,0006105
Std. Dev. =0,01529634
N =11 177

Dynamic portfolio selection: multi-period investment

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Constantly Re-balanced Portfolio (CRP)

a portfolio vector $\mathbf{b} = (b^{(1)}, \dots b^{(d)})$
$b^{(j)}$ gives the proportion of the investor's capital invested in stock j
$\mathbf{b}$ is the constant portfolio vector for each trading day

Evolution

for the first day S_0 denotes the initial capital

$$S_1 = S_0 \sum_{j=1}^{d} b^{(j)} x_1^{(j)} = S_0 \langle \mathbf{b} \, , \, \mathbf{x}_1 \rangle$$

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for the second day, S_1 new initial capital

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with the average growth rate

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Special market process: $\mathbf{X}_1, \mathbf{X}_2, \dots$ is independent and identically distributed (i.i.d.)

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universal portfolio

for dependent market process we can do even better

History

gambling,

gambling, horse racing,

gambling, horse racing, information theory

History

gambling, horse racing, information theory

Algoet

History

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Algoet
Barron

History

gambling, horse racing, information theory

Algoet
Barron
Breiman

History

gambling, horse racing, information theory

Algoet
Barron
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Cover

History

gambling, horse racing, information theory

Algoet
Barron
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Cover
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History

gambling, horse racing, information theory

Algoet
Barron
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Cover
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Latané

Chapter 15 of D. G. Luenberger, *Investment Science*. Oxford University Press, 1998.

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"Fortunately the concepts and the methods of analysis for multiperiod situation build on those of earlier chapters. Internal rate of return, present value, the comparison principle, portfolio design, and lattice and tree valuation all have natural extensions to general situations. But conclusions such as volatility is "bad" or diversification is "good" are no longer universal truths. The story is much more interesting."

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nth day a portfolio strategy $\mathbf{b}_n = \mathbf{b}(\mathbf{x}_1, \dots, \mathbf{x}_{n-1}) = \mathbf{b}(\mathbf{x}_1^{n-1})$

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with the average growth rate

$$W_n(\mathbf{B}) = \frac{1}{n} \sum_{i=1}^{n} \ln \left\langle \mathbf{b}(\mathbf{x}_1^{i-1}) \, , \, \mathbf{x}_i \right\rangle .$$

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$\mathbf{X}_1, \mathbf{X}_2, \ldots$ drawn from the vector valued stationary and ergodic process
log-optimum portfolio $\mathbf{B}^* = \{\mathbf{b}^*(\cdot)\}$

$$\mathbf{E}\{\ln \left\langle \mathbf{b}^*(\mathbf{X}_1^{n-1}), \, \mathbf{X}_n \right\rangle \mid \mathbf{X}_1^{n-1}\} = \max_{\mathbf{b}(\cdot)} \mathbf{E}\{\ln \left\langle \mathbf{b}(\mathbf{X}_1^{n-1}), \, \mathbf{X}_n \right\rangle \mid \mathbf{X}_1^{n-1}\}$$

Optimality

Algoet and Cover (1988): If $S_n^* = S_n(\mathbf{B}^*)$ denotes the capital after day n achieved by a log-optimum portfolio strategy $\mathbf{B}^*$, then for any portfolio strategy $\mathbf{B}$ with capital $S_n = S_n(\mathbf{B})$ and for any stationary ergodic process $\{\mathbf{X}_n\}_{-\infty}^{\infty}$,

$$\limsup_{n\to\infty}\left(\frac{1}{n}\ln S_n - \frac{1}{n}\ln S_n^*\right) \le 0 \quad \text{almost surely}$$

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$$\limsup_{n\to\infty} \left(\frac{1}{n} \ln S_n - \frac{1}{n} \ln S_n^* \right) \le 0 \quad \text{almost surely}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \ln S_n^* = W^* \quad \text{almost surely,}$$

where

$$W^* = \mathbf{E}\left\{ \max_{\mathbf{b}(\cdot)} \mathbf{E}\{\ln \left\langle \mathbf{b}(\mathbf{X}_{-\infty}^{-1}) \, , \, \mathbf{X}_0 \right\rangle \mid \mathbf{X}_{-\infty}^{-1}\} \right\}$$

is the maximal growth rate of any portfolio.

Proof

$$\frac{1}{n}\ln S_n = \frac{1}{n}\sum_{i=1}^{n}\ln\left\langle \mathbf{b}(\mathbf{X}_1^{i-1}),\, \mathbf{X}_i\right\rangle$$

Proof

$$
\begin{aligned}
\frac{1}{n}\ln S_n &= \frac{1}{n}\sum_{i=1}^{n} \ln \left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i \right\rangle \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}\{\ln \left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i \right\rangle \mid \mathbf{X}_1^{i-1}\} \\
&+ \frac{1}{n}\sum_{i=1}^{n} \left(\ln \left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i \right\rangle - \mathbf{E}\{\ln \left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i \right\rangle \mid \mathbf{X}_1^{i-1}\} \right)
\end{aligned}
$$

Proof

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$$= \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}\{\ln\left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\}$$

$$+ \frac{1}{n}\sum_{i=1}^{n}\left(\ln\left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i\right\rangle - \mathbf{E}\{\ln\left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\}\right)$$

and

$$\frac{1}{n}\ln S_n^* = \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}\{\ln\left\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}), \mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\}$$

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Universally consistent portfolio

These limit relations give rise to the following definition:

Definition

A portfolio strategy $\mathbf{B}$ is called **universally consistent with respect to a class $\mathcal{C}$ of stationary and ergodic processes** $\{\mathbf{X}_n\}_{-\infty}^{\infty}$, if for each process in the class,

$$\lim_{n\to\infty} \frac{1}{n} \ln S_n(\mathbf{B}) = W^* \quad \text{almost surely.}$$

$$\mathbf{E}\{\ln\left\langle \mathbf{b}^*(\mathbf{X}_1^{n-1}), \mathbf{X}_n\right\rangle \mid \mathbf{X}_1^{n-1}\} = \max_{\mathbf{b}(\cdot)} \mathbf{E}\{\ln\left\langle \mathbf{b}(\mathbf{X}_1^{n-1}), \mathbf{X}_n\right\rangle \mid \mathbf{X}_1^{n-1}\}$$

Empirical portfolio selection

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fixed integer $k > 0$

$$\mathbf{E}\{\ln \left\langle \mathbf{b}(\mathbf{X}_1^{n-1}), \mathbf{X}_n \right\rangle \mid \mathbf{X}_1^{n-1}\} \approx \mathbf{E}\{\ln \left\langle \mathbf{b}(\mathbf{X}_{n-k}^{n-1}), \mathbf{X}_n \right\rangle \mid \mathbf{X}_{n-k}^{n-1}\}$$

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and

$$\mathbf{b}^*(\mathbf{X}_1^{n-1}) \approx \mathbf{b}_k(\mathbf{X}_{n-k}^{n-1}) = \arg\max_{\mathbf{b}(\cdot)} \mathbf{E}\{\ln \left\langle \mathbf{b}(\mathbf{X}_{n-k}^{n-1}), \mathbf{X}_n \right\rangle \mid \mathbf{X}_{n-k}^{n-1}\}$$

because of stationarity

$$\mathbf{b}_k(\mathbf{x}_1^k) = \arg\max_{\mathbf{b}(\cdot)} \mathbf{E}\{\ln \left\langle \mathbf{b}(\mathbf{X}_{n-k}^{n-1}), \mathbf{X}_n \right\rangle \mid \mathbf{X}_{n-k}^{n-1} = \mathbf{x}_1^k\}$$

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which is the maximization of the regression function

$$m_{\mathbf{b}}(\mathbf{x}_1^k) = \mathbf{E}\{\ln \langle \mathbf{b} \,,\, \mathbf{X}_{k+1} \rangle \mid \mathbf{X}_1^k = \mathbf{x}_1^k\}$$

Regression function

Y real valued
X observation vector

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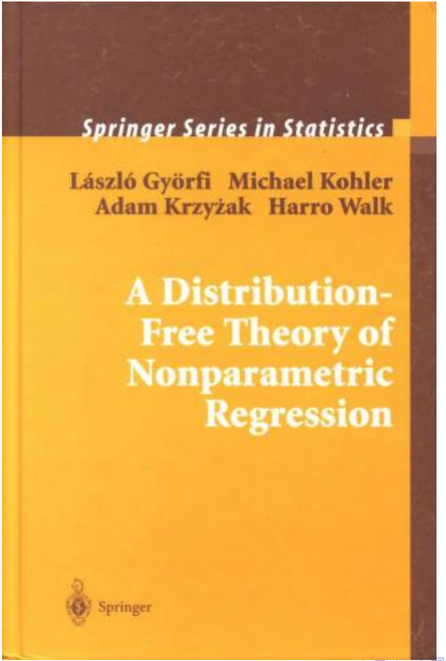
L. Györfi, M. Kohler, A. Krzyzak, H. Walk (2002) *A Distribution-Free Theory of Nonparametric Regression*, Springer-Verlag, New York.


Springer Series in Statistics
László Györfi Michael Kohler
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A Distribution-Free Theory of Nonparametric Regression
Springer

Correspondence

$$X \quad \sim \quad \mathbf{X}_1^k$$

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\begin{aligned}
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Correspondence

$$\begin{aligned} X &\sim \mathbf{X}_1^k \\ Y &\sim \ln \langle \mathbf{b}\,,\, \mathbf{X}_{k+1} \rangle \\ m(x) = \mathbf{E}\{Y \mid X = x\} &\sim m_{\mathbf{b}}(\mathbf{x}_1^k) = \mathbf{E}\{\ln \langle \mathbf{b}\,,\, \mathbf{X}_{k+1} \rangle \mid \mathbf{X}_1^k = \mathbf{x}_1^k\} \end{aligned}$$

Kernel-based portfolio selection

choose the radius $r_{k,\ell} > 0$ such that for any fixed k,

$$\lim_{\ell \to \infty} r_{k,\ell} = 0.$$

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for $n > k + 1$, define the expert $\mathbf{b}^{(k,\ell)}$ by

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where

$$J_n^{(k,\ell)} = \left\{ k < i < n : \|\mathbf{x}_{i-k}^{i-1} - \mathbf{x}_{n-k}^{n-1}\| \le r_{k,\ell} \right\}$$

if the sum is non-void, and $\mathbf{b}_0 = (1/d, \ldots, 1/d)$ otherwise.

Combining elementary portfolios

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$\mathbf{B}^{(k,\ell)} = \{\mathbf{b}^{(k,\ell)}(\cdot)\}$, are called elementary portfolios

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- small k or large $r_{k,\ell}$: large bias
- large k and small $r_{k,\ell}$: few matching, large variance

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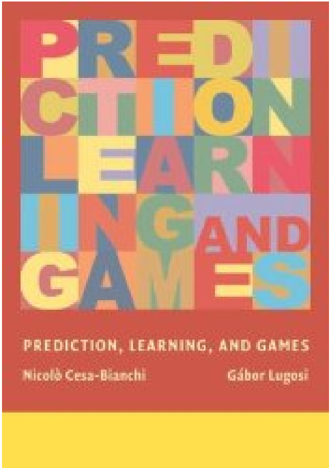
Machine learning: combination of experts

N. Cesa-Bianchi and G. Lugosi, *Prediction, Learning, and Games.* Cambridge University Press, 2006.


PREDICTION LEARNING AND GAMES
PREDICTION, LEARNING, AND GAMES
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$$w_{n,k,\ell} = q_{k,\ell} S_{n-1}(\mathbf{B}^{(k,\ell)})$$

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the aggregated portfolio $\mathbf{b}$:

$$\mathbf{b}_n(\mathbf{x}_1^{n-1}) = \frac{\sum_{k,\ell} w_{n,k,\ell} \mathbf{b}_n^{(k,\ell)}(\mathbf{x}_1^{n-1})}{\sum_{k,\ell} w_{n,k,\ell}}.$$

Theorem

The kernel-based portfolio scheme is universally consistent with respect to the class of all ergodic processes such that $\mathbf{E}\{|\ln X^{(j)}|\} < \infty$, for $j = 1, 2, \ldots, d$.

L. Györfi, G. Lugosi, F. Udina (2006) "Nonparametric kernel-based sequential investment strategies", *Mathematical Finance*, 16, pp. 337-357
www.szit.bme.hu/~gyorfi/kernel.pdf

Proof

$$S_n(\mathbf{B}) = \prod_{i=1}^{n} \left\langle \mathbf{b}_i(\mathbf{x}_1^{i-1}) \, , \, \mathbf{x}_i \right\rangle$$

Proof

$$
\begin{aligned}
S_n(\mathbf{B}) &= \prod_{i=1}^{n} \left\langle \mathbf{b}_i(\mathbf{x}_1^{i-1}) \, , \, \mathbf{x}_i \right\rangle \\
&= \prod_{i=1}^{n} \frac{\sum_{k,\ell} w_{i,k,\ell} \left\langle \mathbf{b}_i^{(k,\ell)}(\mathbf{x}_1^{i-1}) \, , \, \mathbf{x}_i \right\rangle}{\sum_{k,\ell} w_{i,k,\ell}}
\end{aligned}
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&= \prod_{i=1}^{n} \frac{\sum_{k,\ell} q_{k,\ell} S_{i-1}(\mathbf{B}^{(k,\ell)}) \left\langle \mathbf{b}_i^{(k,\ell)}(\mathbf{x}_1^{i-1}) \,,\, \mathbf{x}_i \right\rangle}{\sum_{k,\ell} q_{k,\ell} S_{i-1}(\mathbf{B}^{(k,\ell)})}
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&= \prod_{i=1}^{n} \frac{\sum_{k,\ell} q_{k,\ell} S_{i-1}(\mathbf{B}^{(k,\ell)}) \left\langle \mathbf{b}_i^{(k,\ell)}(\mathbf{x}_1^{i-1}) \, , \, \mathbf{x}_i \right\rangle}{\sum_{k,\ell} q_{k,\ell} S_{i-1}(\mathbf{B}^{(k,\ell)})} \\
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&= \sum_{k,\ell} q_{k,\ell} S_n(\mathbf{B}^{(k,\ell)}),
\end{aligned}
$$

Equivalent form of aggregation

The strategy $\mathbf{B}$ then arises from weighing the elementary portfolio strategies $\mathbf{B}^{(k,\ell)} = \{\mathbf{b}_n^{(k,\ell)}\}$ such that the investor's capital becomes

$$S_n(\mathbf{B}) = \sum_{k,\ell} q_{k,\ell} S_n(\mathbf{B}^{(k,\ell)}).$$

We have to prove that

$$\liminf_{n\to\infty} W_n(\mathbf{B}) = \liminf_{n\to\infty} \frac{1}{n} \ln S_n(\mathbf{B}) \geq W^* \quad \text{a.s.}$$

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&= \sup_{k,\ell} \left(W_n(\mathbf{B}^{(k,\ell)}) + \frac{\ln q_{k,\ell}}{n} \right).
\end{aligned}$$

Thus

$$\liminf_{n\to\infty} W_n(\mathbf{B}) \quad \geq \quad \liminf_{n\to\infty} \sup_{k,\ell} \left(W_n(\mathbf{B}^{(k,\ell)}) + \frac{\ln q_{k,\ell}}{n} \right)$$

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&= \sup_{k,\ell} \liminf_{n\to\infty} W_n(\mathbf{B}^{(k,\ell)}) \\
&= \sup_{k,\ell} \epsilon_{k,\ell}
\end{aligned}$$

Because of $\lim_{\ell\to\infty} r_{k,\ell} = 0$, we have that

$$\sup_{k,\ell} \epsilon_{k,\ell} = \lim_{k\to\infty} \lim_{l\to\infty} \epsilon_{k,\ell} = W^*.$$

Semi-log-optimal (mean-variance) portfolio

empirical log-optimal:

$$\mathbf{b}^{(k,\ell)}(\mathbf{x}_1^{n-1}) = \arg\max_{\mathbf{b}} \sum_{i \in J_n^{(k,\ell)}} \ln \langle \mathbf{b} \, , \, \mathbf{x}_i \rangle$$

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Taylor expansion: $\ln z \approx h(z) = z - 1 - \frac{1}{2}(z-1)^2$

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Connection to the Markowitz theory

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L. Györfi, A. Urbán, I. Vajda (2007) "Kernel-based semi-log-optimal portfolio selection strategies", *International Journal of Theoretical and Applied Finance*, 10, pp. 505-516. www.szit.bme.hu/∼gyorfi/semi.pdf

Conditions of the model:

Assume that

- the assets are arbitrarily divisible,
- the assets are available in unbounded quantities at the current price at any given trading period,
- there are no transaction costs,
- the behavior of the market is not affected by the actions of the investor using the strategy under investigation.

NYSE data sets

At www.szit.bme.hu/~oti/portfolio there are two benchmark data set from NYSE:

- The first data set consists of daily data of 36 stocks with length 24 years (1962–1985)
- The second data set contains 19 stocks and has length 45 years (1962–2006)

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Our experiment is on the second data set

Experiments on average annual yields (AAY) for kernel based experts

Kernel based semi-log-optimal portfolio selection with

Experiments on average annual yields (AAY) for kernel based experts

Kernel based semi-log-optimal portfolio selection with finite array of size $K \times L$ such that $K = 5$ and $L = 10$. Choose the uniform distribution $\{q_{k,\ell}\} = 1/(KL)$ over the experts in use.

Experiments on average annual yields (AAY) for kernel based experts

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$$r_{k,l}^2 = 0.0002 \cdot d \cdot k + 0.00002 \cdot d \cdot k \cdot \ell,$$

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AAY of kernel based semi-log-optimal portfolio is 31%
MORRIS had the best AAY, 20%
the BCRP had average AAY 21%

The average annual yields of the of the kernel based experts.

k / ℓ	1	2	3	4	5
1	31%	30%	24%	21%	26%
2	34%	31%	27%	25%	22%
3	35%	29%	26%	24%	23%
4	35%	30%	30%	32%	27%
5	34%	29%	33%	24%	24%
6	35%	29%	28%	24%	27%
7	33%	29%	32%	23%	23%
8	34%	33%	30%	21%	24%
9	37%	33%	28%	19%	21%
10	34%	29%	26%	20%	24%

Experiments on average annual yields for nearest neighbor based experts

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Experts are indexed by $k = 1 \dots 5$ in columns and
$\ell = 50, 100, \dots, 500$ in rows, where ℓ is the number of nearest neighbors.

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Experts are indexed by $k = 1 \ldots 5$ in columns and $\ell = 50, 100, \ldots, 500$ in rows, where ℓ is the number of nearest neighbors.
The average annual yield of nearest neighbor portfolio is 35%
Comparing the tables, one can conclude that the nearest neighbor strategy is more robust.

The average annual yields of the nearest neighbor based experts.

k / ℓ	1	2	3	4	5
50	31%	33%	28%	24%	35%
100	33%	32%	25%	29%	28%
150	38%	33%	26%	32%	27%
200	38%	28%	32%	32%	24%
250	37%	31%	37%	28%	26%
300	41%	35%	35%	30%	29%
350	39%	36%	31%	34%	32%
400	39%	35%	33%	32%	35%
450	39%	34%	34%	35%	37%
500	42%	36%	33%	38%	35%